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TRANSMISSION CIRCUITS FOR TELEPHONIC COMMUNICATION

Methods of Analysis and Design

By

K. S. JOHNSON

Member of Technical Staff, Bell Telephone Laboratories, Incorporated

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FOREWORD

Since 1919 there has been conducted in the Bell Telephone Laboratories a program of out-of-hour courses through which the members of its technical staff could obtain the benefit of the technical information and experience of its experts.

Courses are arranged in accordance with the needs of the staff and as advances are made in the art of electrical communication. They are given in the hours immediately preceding or following regular business hours. Registration is optional and the instructors, all of whom are members of the technical staff, serve without extra compensation. The text for each course is prepared by the instructor from the technical files and experimental data of the Laboratories. The material at his disposal is the cumulation of the work of his colleagues and predecessors as well as of himself. In the preparation and coordination of a text suited to the particular needs of the course each instructor chooses his own material, the order of presentation, and methods of proof.

The present volume is the text developed for one of these special courses. It was originally printed privately for use within the Laboratories. The character of the material, however, and the method of presentation have attracted so much favorable comment and resulted in so many requests for a wider circulation that it is now made available through regular channels of publication.

BELL TELEPHONE LABORATORIES, Incorporated.

March, 1925.

PREFACE

The evolution of this volume of notes has extended through some fifteen years. During the early part of that time the writer was engaged in transmission engineering in what was then the Transmission and Protection Department of the Engineering Department of the American Telephone and Telegraph Company. The first contributions to his present point of view as to telephonic transmission were therefore obtained under the leadership of Dr. F. B. Jewett, now Vice-President of the Western Electric Company, Inc., and Mr. O. B. Blackwell, now Transmission Development Engineer of the American Telephone and Telegraph Company, who were successively in charge of that department.

These notes took their first form during 1918–1919 when the writer conducted one of the special courses for other members of the Engineering Department of the Western Electric Company, who were interested in an analytical study of telephonic transmission. During successive repetitions of that course these notes have reached the more complete and comprehensive form of the present volume. They were first publicly used in a series of lectures given at Harvard University as Course 220 during the winter of 1921–22 at the invitation of the Department of Electrical Engineering of that institution.

The material of the notes is intended to cover the general theory and principles which are applicable to the development and design of circuits and lines for association with such telephonic instruments as transmitters, receivers, and vacuum tubes. The detailed characteristics and methods of design for these instruments are not here considered. In the program of technical courses in the Engineering Department these instruments are covered by other special courses for which, however, much of the material here presented is prerequisite.

The text assumes on the part of the reader a familiarity with the elements of alternating current theory and with the operations of differential calculus. The student should also be familiar with the operations and notations of the complex-quantity method of representing alternating functions. For the convenience of those who wish a rapid review of this subject, condensed formulae illustrating a satisfactory procedure for handling complex quantities are given in Appendix B.

As previously stated, the material at the writer's disposal in developing the text has been the accumulation of the work of his colleagues and predecessors as well as of himself. In addition, however, to this general recognition of the contributions of others, and the influence upon himself of the scientific atmosphere in which the past fifteen years have been spent, the writer would particularly like to acknowledge the influence of the work and methods of Dr. G. A. Campbell and the importance of the more recent contributions of Dr. O. J. Zobel. He also wishes to acknowledge his great indebtedness to those of his associates who have assisted in the preparation of the manuscript.

K. S. J.

New York City,
June, 1924.

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CHAPTER I

FUNDAMENTAL FACTS ABOUT THE VOICE, EAR, TRANSMITTER AND RECEIVER

1.0 In the analysis and design of circuits for telephonic transmission. it is necessary to know or assume certain facts regarding the action of the human voice and ear as well as regarding the characteristics of the telephone instruments which are to be employed in such circuits.

1.1 Voice and Ear.—It is a well established proposition that any complex wave in a steady state condition, such as that of a voiced sound, can be analyzed into an infinite number of pure sinusoidal waves of different amplitudes. While an infinite number of such waves are theoretically required to represent exactly such a complex wave, only a comparatively small number are required to represent the wave to a degree of accuracy sufficient for practical engineering purposes. Therefore, when any syllable is enunciated a very large number of different vibrations are set up in the air, each vibration of which adds something to the interpretability, pitch and timbre of the syllable. The human voice contains important sound vibrations or frequencies ranging from about 100 or 200 cycles per second to frequencies as high as 8,000 or 10,000 cycles per second. Any particular sound associated with a syllabic significance is made up of a range of frequencies lying somewhere in this frequency range. The most important range of frequencies probably lies between 250 cycles and 2,500 cycles.

All discussions and computations, except when noted to the contrary, assume that the steady state—as opposed to the transient state—of the electrical system has been attained. Since the transient state exists for only a very small period, this assumption is permissible in a large percentage of all telephonic transmission problems.

We will now examine in some detail the characteristics of the human ear. It is a well-known fact that the ear can hear some sounds more easily than others. A considerable amount of research has been done lately in this field by Fletcher, Wegel, and others. Two curves known as the *threshold of audibility* and the *threshold of feeling* have been presented and discussed by Wegel and Fletcher.[*] These curves show that if a sound vibration is generated within the audible range of frequency, there is a corresponding minimum amplitude of pressure change (usually measured in dynes per square centimeter change of pressure on the ear) which the ear can just hear. As this amplitude of pressure change increases, that is, as the sound becomes progressively louder, a maximum amplitude is reached at which the ear not only *hears* but physically *feels* the sound as a prickling sensation or pain. In a similar manner there exists a minimum amplitude and a maximum amplitude of change of pressure on the ear for each frequency within the audible frequency range. Between these pressure amplitudes the ear can hear. It is customary to call the curve defined by the pressure amplitudes of minimum audibility the *threshold of audibility* and that defined by the pressure amplitudes of feeling the *threshold of feeling*. Such curves obtained for the normal ear, as plotted by Wegel, are reproduced in Fig. 1.

---

[*] "The Physical Examination of Hearing and Binaural Aids for the Deaf," R. L. Wegel, *National Academy of Sciences*, July 15, 1922, pp. 155–160.

"Physical Measurements of Audition," H. Fletcher, *Journal of the Franklin Institute*, Sept., 1923, pp. 289–326.

"The Frequency-Sensitivity of Normal Ears," H. Fletcher and R. L. Wegel, *Physical Review* Vol. 19, No. 6, pp. 553–565.

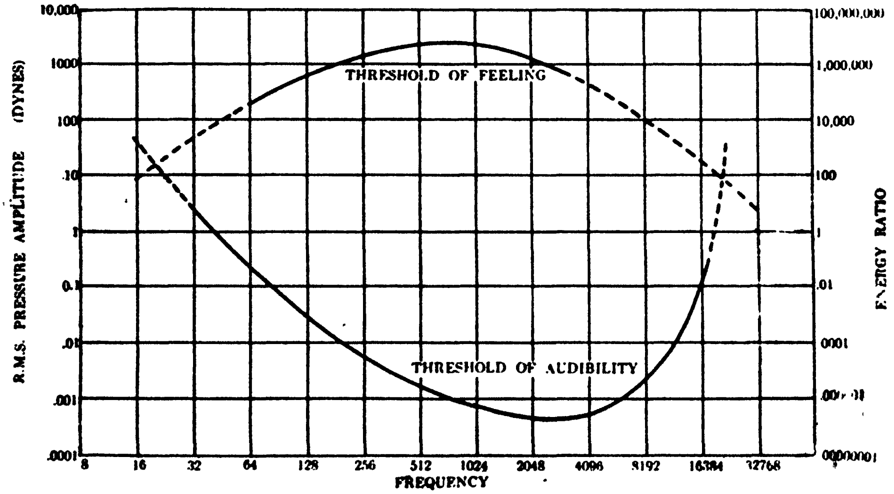

Fig. 1. Curves enclosing the auditory sensation area.

It is seen that throughout the most important part of the voice-frequency range the change of pressure range for hearing is from .001 dyne to 1,000 dynes per square centimeter.

At 60 cycles the sensation of feeling for strong vibrations can be described by the word "flutter." This is more marked at lower frequencies, e. g. 20 cycles, where only a small intensity is necessary in order that the vibration be felt and it becomes difficult to distinguish between the senses of feeling and hearing. A similar condition is approached at very high frequencies.

For a given frequency of vibration there is, in accordance with *Weber's Law*, a definite minimum increment of loudness which the average ear can detect. Knudsen finds that this minimum increment of the average normal ear is practically constant between 100 and 4,000 cycles. About 270 of these loudness increments exist between the audibility limits of the normal ear at 1,000 cycles. Similarly there is a definite minimum increment of frequency which the normal ear can detect. For example, Fletcher finds that the number of tones that are perceivable of being different in pitch along a 10 dyne amplitude pressure line of loudness is about 1,300. Therefore, it appears that the audibility area bounded by the two threshold curves can be divided up into small elementary squares, all points of which in any one square represent the *same* sound to the ear, although every point in it is different from every other in pitch or loudness or in both. On this basis of reasoning it has been estimated that there are about 300,000 such squares and therefore that the ear can detect 300,000 pure tones differing in pitch and loudness. The number of complex tones appearing different to the ear is probably far greater than this.

In determining the relative importance of the various frequencies in the human voice, and for similar investigations, a very high quality telephone system is employed. Over such a system one can hear as well as by direct transmission through the air. Such a system is evidently very useful in experimental work but since it involves the use of very costly apparatus, such as condenser transmitters, amplifiers, equalizers and highly damped receivers, it is not practical for use in commercial systems. The use of such a system however makes it possible, among other things, to determine the relative difficulty of transmitting various sounds.

Generally speaking it is found that consonants are harder to hear than are equally loud vowels. There are, however, a few exceptions to this rule. For example, *e* is one of the most difficult letters to interpret while *l*, *r* and *ng* are among the easiest speech sounds to recognize at average intensities. *Th*, *f* and *v* are difficult to hear regardless of the intensity and account for over 50 per cent of all the errors occurring in a commercial telephone system while *z*, *h* and *s* are especially difficult to understand at weak intensities. It is also found that *i*, *ou*, *er*, and *o* are missed less than 10 per cent of the time even with very weak intensities.

Investigations have also been made, using the high quality telephone system, to determine the relative importance of different frequency bands in the voice-frequency range. This was done by inserting high, low or band pass filters in the circuit which normally transmitted all frequencies occurring in the human voice, thereby causing the received speech to be made up of a limited range of frequencies. There are two important facts that are brought out by this method of analysis. One is that the lower frequencies of speech furnish the greater part of the acoustic energy and the other is that the higher frequencies furnish the greater part of the *articulation*—a term that is practically synonymous with interpretability. It is found that the average energy of syllabic speech is reduced by 60 per cent when frequencies below 500 cycles are eliminated while the articulation is reduced by only 2 per cent. As a converse example, eliminating all frequencies above 1,500 cycles reduces the energy by about 10 per cent while the articulation is reduced by 35 per cent. It is also interesting to observe that syllabic articulation is reduced by the same amount—about 15 per cent—when frequencies either above 3,000 cycles or below 1,000 cycles are eliminated. Curves, obtained by Fletcher, showing the energy and articulation of syllabic speech, are reproduced in Fig. 2.

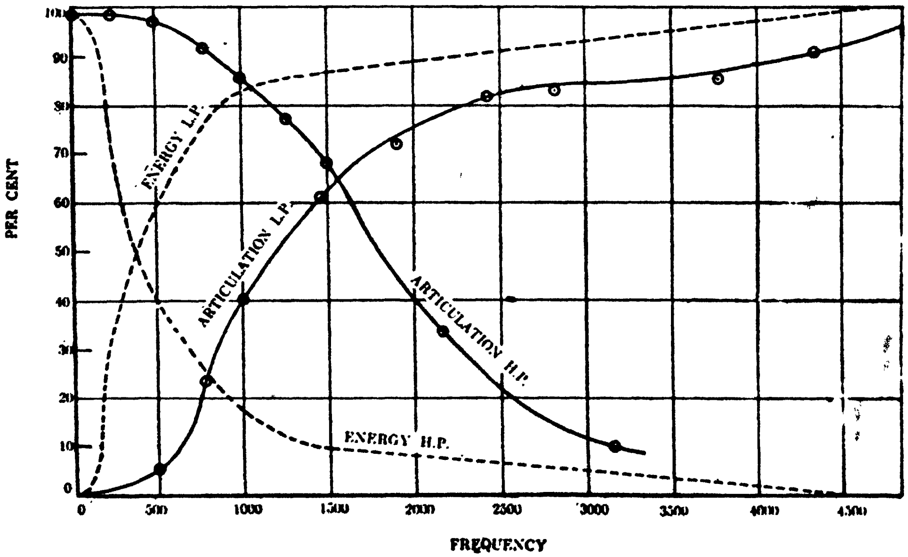

FIG. 2. Effect upon the articulation and the energy of speech due to the elimination of certain frequency regions.

These curves were obtained by inserting low pass filters (indicated by L.P.) or high pass filters (indicated by H.P.) in the high quality system previously referred to. For example the point (2,000, 40) indicates that when a high pass filter was inserted, the cut-off of which was 2,000 cycles, the articulation was 40 per cent. The point (500, 60) indicates that when a low pass filter, the cut-off frequency of which was 500 cycles, was inserted, 60 per cent of the total voice energy remained.

L, *i*, and *ē* can be transmitted with an error of less than 3 per cent by transmitting approximately half of the voice range of frequencies. On the other hand, the short vowels *u*, *o* and *e* have important characteristics carried by frequencies below 1,000 cycles. If this frequency range is eliminated, 20 per cent error or 80 per cent articulation occurs for these sounds while removing frequencies above 2,000 cycles causes practically no error. The fricative consonants *s*, *z* and *th* need for complete enunciation frequencies above 5,000 cycles. If these higher frequencies are eliminated, the effect is noticeable although all frequencies below 1,500 cycles can be removed without effect.

For a given telephone system, assuming the speech is clearly enunciated into the transmitter, there is a definite intensity or loudness in the received speech which gives maximum intelligibility—any decrease or increase of intensity resulting in a *decrease* in intelligibility. The decrease which is produced by a large increase of intensity is partially due to the fact that the response of the ear deviates more and more from the condition of being proportional to the sound stimulus striking it. However the optimum intensity increases as extraneous noise is introduced into the system in increasing amounts.

Heretofore, 800 cycles has been considered the best representative single frequency of the voice and a large number of computations of telephone engineering problems have been made at this frequency. In the light of recent investigations, however, it appears that if a single frequency is to be used, computations on commercial telephone systems will more nearly check with the results of voice tests if a single sinusoidal frequency of 1,000 or 1,100 cycles is assumed. It is evident, however, that inasmuch as intelligible speech cannot be transmitted by means of any single sinusoidal wave it is necessary, if a high degree of accuracy is desired, to make computations at several different frequencies and weight the results in accordance with their observed importance in contributing to the transmission of intelligence.

1.2 Telephone Instruments.

1.21 Transmitters.—In circuit analysis and design it is frequently necessary to make certain assumptions regarding the characteristics of the instruments used for converting sound power into electrical power and vice versa. In the case of the transmitter it is ordinarily assumed that the instrument acts as an a-c. generator provided a given amount of direct current flows through it or a given magnetic flux exists—if the instrument is of the magnetic type. It is, furthermore, assumed that for a definite agitation of the transmitter there will be effectively a definite internally generated electromotive force acting in series with the impedance of the transmitter. In other words, for practically all circuit problems *a transmitter can be regarded as a source of constant e.m.f. acting in series with the impedance, Z_T, of the transmitter.* This e.m.f. moreover is not affected by changing the manner of supplying the direct current—provided, of course, that the latter is kept constant. These statements appear to be justified by a considerable amount of both theoretical and experimental investigation.

One of the important characteristics of the carbon button type of transmitter—the type in common use—is its impedance characteristic. It has been found that this is practically *pure resistance* (i.e., it has no reactance). Investigation has furthermore shown that the *effective resistance* of a transmitter is not, in general, equal to its *talking resistance*—as measured by a d-c. voltmeter and ammeter. *The effective or a-c. resistance, R, of any device is defined as the ratio of the power absorbed by the device to the square of the current flowing through it.* It is frequently assumed that the alternating-current or effective resistance of a carbon button transmitter is approximately 80 per cent of its apparent talking resistance—a figure which holds closely for a considerable number of typical cases. Although the effective resistance of a transmitter varies with the direct current flowing through it and with the amount of agitation, it is, however, essentially independent of the a-c. load to which the transmitter may be connected. In certain types of transmission problems it is frequently necessary to determine the variation in the transmitter resistance with the change of direct current, but in such cases it is usually assumed that the transmitter is subjected to a definite amount of agitation.

A fairly accurate method of measuring the effective resistance of a carbon button transmitter is by means of a circuit which makes it possible to determine when the voltage across a known resistance, to which the transmitter is connected, is one-half of the e.m.f. generated by the transmitter on open circuit. A simplified sketch of this circuit is shown in Fig. 3. The operation of this circuit is as follows. With the transmitter agitated by the voice or some other suitable source of energy, with the resistance *R* made infinite and with the voltmeter *V* across the terminals *A–C*, one-half of the total e.m.f. *E* generated by the transmitter is evidently measured by the voltmeter. This is, of course, on the assumption that the impedance of the retardation coil, as well as that of the two 5,000 ohm resistances, is very large in comparison with the transmitter resistance and that the 2 mf. condenser is of sufficiently large capacity so that its impedance is negligibly small. Then, with the voltmeter connected across the terminals *B–C*, *R* is varied until the voltmeter reads the same as in the preceding case. The corresponding value of *R* will evidently be equal to the a-c. resistance of the transmitter—on the assumption previously made that the carbon button transmitter can be regarded as a source of constant e.m.f. *E* acting through the a-c. resistance of the transmitter.

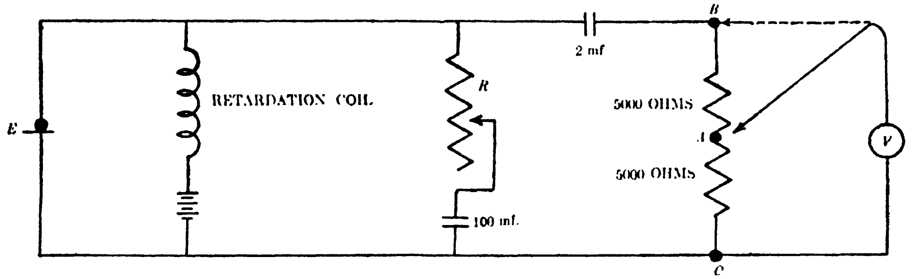

FIG. 3. Circuit for measuring the a-c. resistance of a transmitter.

1.22 Receivers.—*An ideal telephone receiver would, at any frequency, convert into sound all the electrical power which it absorbs.* In the non-vibrating or damped condition the impedance of the ideal receiver would be pure inductive or pure capacitative reactance. This condition would also hold for the ideal receiver if vibrating in a vacuum.*

Actual receivers do not, of course, convert into sound all of the power absorbed by them. The efficiency of receivers also varies to some extent with the amount of electrical power put into them. However, for most practical purposes it is safe to assume that the efficiency of the receiver is more or less independent of the electrical power absorbed, and that for maximum electrical power absorbed maximum sound output is produced.

Present types of permanent magnet receivers have considerable reactance, and the phase angle of their damped impedance is approaching, fairly closely, $\underline{/90^\circ}$. As an example of this it is interesting to note that the hand receiver which was commonly employed in subscribers' stations until about 1912 has a damped impedance whose phase angle is approximately $\underline{/40^\circ}$. The receiver which at present is standard in substation sets has an impedance whose phase angle is roughly $/\underline{50^\circ}$, while the present standard type of operators' receiver has an impedance whose phase angle is more nearly $/\underline{60^\circ}$. Electromagnet receivers, or receivers which have no permanent magnet in them, have a still higher phase angle, which often approximates $/\underline{70^\circ}$. As will be seen later, this high phase angle makes it extremely difficult to get the maximum possible power into the receiver, except over a small range of frequencies.

---

* For a complete discussion of the theory of the telephone receiver, see "Electrical Vibration Instruments," by A. E. Kennelly, and "Theory of Magneto-Mechanical Systems as Applied to Telephone Receivers and Similar Structures," by R. L. Wegel, *Journal* A. I. E. E., Vol. 40, Oct., 1921, pp. 791–802. For a more condensed discussion, see Chapter II of "Radio Communication" by J. Mills.

An assumption frequently made in receiver design is that the absolute magnitude of the impedance of the receiver may be varied over wide limits, by varying the number of turns in its winding, without affecting the phase angle. This result can be accomplished very closely provided the receiver is wound with the proper number of turns—always keeping the winding volume or cross-section the same. This is equivalent to assuming that the receiver impedance can be varied in exactly the same way as could be done if it were associated with an ideal transformer of the desired impedance ratio. This assumption fails to be justified only where very small gauges of wire (such as No. 39 or No. 40 A.W.G.) are employed, in which case the ratio of copper to that of insulation becomes so small that the d-c. and hence the effective resistance of the receiver goes up faster, relatively, than does its inductance, thereby resulting in a decreased phase angle.

CHAPTER II

MEASURES OF TELEPHONIC EFFICIENCIES

2.0 In any telephone system there are two factors which determine the *intelligibility* of the system—or its capability for transmitting the meaning of the connected and organized speech sounds of conversation. These factors are (1) the volume or acoustic power transmitted over the system and (2) the quality or the faithfulness of reproduction of the speech sounds. In order then to determine quantitatively how any system compares with any other system, it is necessary to have quantitative means for expressing (1) the *volume efficiency* of the system and (2) its *articulation efficiency.*

2.1 Volume Efficiency.—Largely on account of the radical differences, which evidently exist between the problems and requirements of power and telephonic transmission, the same terms used for each field have grown to have different meanings. In telephone work *volume efficiency* is usually expressed as a *loss below* or as a *gain above* some standard condition and as a result the terms efficiency and loss—or gain—have essentially the same meaning.

Three different units have been used to express telephonic volume efficiencies: (1) the *natural attenuation unit,* also designated as a *napier* or as a *hyp;* (2) the *mile of standard cable* or simply a *mile,* and (3) the so-called *transmission unit* or TU. Although these three types of units are defined in radically different ways it will be noted from what follows that, in the last analysis, each is simply a logarithmic measure of the ratio of one power to another and consequently that the units differ from each other, basically, only in magnitude. The desirability of employing logarithmic units is based on *Weber's Law,* which states essentially that any two sounds of different loudness but of the same nature can just be distinguished as different from each other provided the intensity of one sound bears a certain *ratio* to the intensity of the other.

2.11 The Natural Attenuation Unit, Napier or Hyp.—Since any change of current can always be expressed in exponential form, perhaps one of the most obvious measures of a current ratio is the exponent to which the natural base ϵ has to be raised to give this ratio, thus

$$I_1 = I_2\epsilon^{\Theta} \qquad \text{or} \qquad \Theta = \log_\epsilon \frac{I_1}{I_2} \tag{1}$$

where I_1 is the value of the current nearer the source of e.m.f. and I_2 is the value of the current more remote. Each of these currents is assumed to be flowing into the same or equal impedances. In the above equation, Θ will in general be a complex quantity—of the form $\alpha + j\beta$. As a rule, however, we are only interested in the real part of Θ, since this is the portion of Θ which determines the relative *magnitude* of the two currents I_1 and I_2. For example, from the relation given in equation (37) of Appendix B

$$\epsilon^{\Theta} = \epsilon^{\alpha+j\beta} = \frac{I_1}{I_2}; \qquad \epsilon^{\alpha} = \left|\epsilon^{\alpha+j\beta}\right| = \left|\frac{I_1}{I_2}\right|$$

or

$$N_A \equiv \alpha = \log_{\epsilon}\left|\frac{I_1}{I_2}\right| = 2.3026 \log_{10}\left|\frac{I_1}{I_2}\right| \tag{2}$$

where the vertical bars are used to signify the absolute or scalar magnitude of the quantities they enclose and the three parallel bars (≡) are used to signify *equals by definition*. In other words, N_A is stated to be equal to α, and is the number of attenuation units corresponding to the given current ratio. The value of α in (2) is then a measure of the ratio of the absolute values of two currents—provided they flow into the same or equal impedances—and is expressed directly in *napiers* or *hyps*.

If W_1 and W_2 represent the corresponding powers produced by the flow of I_1 and I_2 into the same resistances, it is apparent that equation (2) may be written

$$N_A = 1.151 \log_{10}\frac{W_1}{W_2} \tag{3}$$

2.12 The Mile of Standard Cable.—Largely on account of the inconvenient size of the napier, another unit has been used quite generally in the past, whose size is a great deal more convenient in the handling of most telephone transmission problems. The name of this unit, which is known as a *mile of standard cable*, or simply a *mile*, is also fairly significant since the number of miles or the distance over which a person can talk with satisfactory volume is, when other things are equal, a logical measure of the efficiency of any communication system.

In order that we may obtain a clearer idea of what is meant by the loss of a mile of standard cable, it is first necessary to get a more concrete picture of the definition of *standard cable* and a knowledge of its propagation characteristics. *Standard cable is defined as a cable having uniformly distributed resistance of* 88 *ohms per loop mile and uniformly distributed shunt capacitance of* .054 *microfarad per mile.* Its series inductance and shunt leakance are assumed to be zero.* The mile of standard cable is based upon the constant percentage decrease in current (or in power) per mile which takes place when an e.m.f. of any frequency, such as f cycles ($f = \omega/2\pi = 5{,}000/2\pi$), is impressed at one end of an infinite length of such a cable. From the definition of the term *propagation constant*, P (see Sect. 11.11), the ratio of currents, at any two points separated by $N_{\text{M.S.C.}}$ miles of standard cable, in an infinite length of such a structure, is

$$\frac{I_1}{I_2} = \epsilon^{PN_{\text{M.S.C.}}} \equiv \epsilon^{(A+jB)N_{\text{M.S.C.}}}$$

or

$$\left|\frac{I_1}{I_2}\right| = \epsilon^{AN_{\text{M.S.C.}}} \tag{4}$$

where I_2 is the current at the point more remote from the source of e.m.f. and where the vertical bars are, as before, used to indicate the absolute magnitude of the quantity enclosed thereby. A is the real portion of the propagation constant or—in the case of a symmetrical structure—is the attenuation constant and at 796.4 cycles ($\omega = 5{,}000$) its value for standard cable may be shown—from formula (29) of Chapter XII—to be equal to .109 napier. From the above, we then have

$$.109 \times N_{\text{M.S.C.}} = \log_\epsilon\left|\frac{I_1}{I_2}\right|$$

or

$$N_{\text{M.S.C.}} = \frac{2.3026}{.109}\log_{10}\left|\frac{I_1}{I_2}\right| = 21.13 \log_{10}\left|\frac{I_1}{I_2}\right| \tag{5}$$

This last expression gives us then the number of miles of standard cable ($N_{\text{M.S.C.}}$) corresponding to any ratio of currents, provided the latter flow into equal impedances. Comparing this expression with that given in (2) it is seen that the unit known as a mile of standard cable is only about 1/10 of the size of the napier.

If W_1 and W_2 represent the corresponding powers, we evidently have

$$N_{\text{M.S.C.}} = 10.56 \log_{10}\left|\frac{W_1}{W_2}\right| \tag{6}$$

Consequently, if we have two conditions, in which the ratio of the received powers is W_1/W_2, equation (6) gives the corresponding number of miles gain or loss—depending upon whether W_1 is greater or less than W_2—

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* In England it has been customary to assume an inductance of .001 henry per loop mile—which assumption reduces the attenuation constant of the cable by approximately 3 per cent.

caused by the change from one condition to the other. The relative size of the mile of standard cable and the various other units is given in Table II. It may be shown from the above formulae, that for small changes, each per cent decrease in current corresponds to a loss of approximately .1 mile of standard cable so that a 10 per cent change in current, for example, corresponds to approximately one mile of standard cable.

2.13 New Telephone Transmission Unit (TU).—Due to the objection that the attenuation constant of standard cable varies with frequency and also to the fact that the use of a frequency of 796.4 cycles ($\omega = 5{,}000$) is not universally agreed upon, there has recently been adopted a new transmission unit which (1) does not vary with frequency, (2) is of a convenient size or magnitude—avoiding one of the chief objections to the napier or natural attenuation unit—and (3) has a simple physical significance so that its eventual universal adoption would appear probable.

This new transmission unit (referred to for the sake of convenience as a TU) *is defined by the relation*

$$N_{TU} = 10 \log_{10} (W_1/W_2) \tag{7}$$

in which N_{TU} *is the number of* TU *by which any two powers* W_1 *and* W_2 *are said to differ.*

The relation between the TU and various power ratios is shown in Table I.

TABLE I

No. of Units, N_{TU}	Approximate Power Ratio	Exact Power Ratio
3	2	1.995
4	2.5	2.512
6	4	3.981
7	5	5.012
9	8	7.943
10	10	10.000

The relative sizes of the various transmission units which are in use are given in Table II.

TABLE II

RELATIONS BETWEEN VARIOUS TYPES OF TRANSMISSION UNITS

Multiply	by	to obtain
miles	0.947	TU
miles	0.109	napiers
napiers	9.175	miles
napiers	8.686	TU
TU	1.056	miles
TU	0.115	napiers

2.2 Articulation Efficiency.—It is evidently of no use to have a telephone system over which a large volume of sound can readily be transmitted unless at the same time the quality of the sound transmitted is also up to a certain grade. It is of course equally undesirable to have a system in which the quality of reproduction is very high but in which the volume of sound transmitted is so low as to be unintelligible. In other words, in order to have a satisfactory system for the transmission of intelligence we must have one which reproduces the initial sound waves with a sufficiently loud volume and at the same time one that reproduces them with a high degree of faithfulness.

The ability of a telephone system to transmit the meaning of connected and organized speech sounds is called the *intelligibility* of the system and is measured by the percentage of total ideas which are successfully conveyed. It is evident that the intelligibility of a system varies with the volume efficiency, the frequency distortion, and room or line noise. It is, moreover, a difficult quantity to measure quantitatively and consequently the intelligibility of a system is obtained indirectly by means of what are called *articulation tests*. The *articulation* of a system is the capability of the system for transmitting *detached* speech sounds and is measured by the percentage of total sounds spoken which are correctly received. In articulation tests, speech sounds are grouped into meaningless monosyllables. Lists of these monosyllables are then read aloud and listeners over the system under test write what sounds they think were spoken. In special high quality systems which have been constructed for laboratory investigations (which systems are highly efficient over a frequency range from 80 to 8,000 cycles) it is possible to obtain an articulation of as high as 98 per cent.

Although articulation tests were proposed in 1910, by Campbell,* it is only within the last five years or so that the technique of making articulation tests has been sufficiently well perfected so that the results obtained can be regarded as reliable. To-day, when any new piece of telephone transmission apparatus is developed, it is customary to make both articulation and volume tests on it and to determine the suitability of the newly developed apparatus by a proper weighting of the results of these tests.

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* "Telephonic Intelligibility," G. A. Campbell, *Philosophical Magazine and Journal of Science*, Vol. 19 (1910), pp. 152–159.

CHAPTER III

CIRCUITS INVOLVING SIMPLY TRANSMITTERS AND RECEIVER

3.0 The simplest telephone circuit which we can form consists of a transmitter directly connected to a receiver. Such a circuit will evidently permit conversation in one direction only. Let us consider such a circuit and determine how it should be designed in order to attain the maximum possible efficiency.

3.1 Simple One-Way Transmission—Negligible Length of Line. --Assuming such a circuit as is shown in Fig. 1 and keeping in mind the assumptions which have previously been made regarding the characteristics of a transmitter, it is evident that the alternating current flowing in the receiver of the circuit is

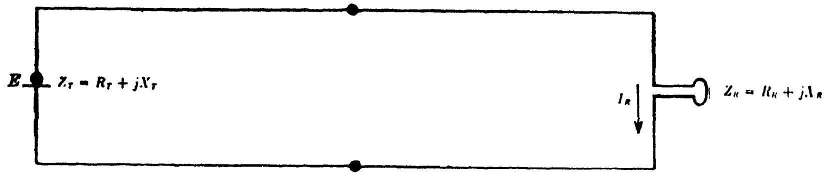

FIG. 1. Simple series circuit, with transmitter directly connected to receiver.

$$I_R = \frac{E}{Z_T + Z_R} = \frac{E}{(R_T + jX_T) + (R_R + jX_R)} \tag{1}$$

where R_T and X_T are respectively the effective resistance and reactance components of the transmitter impedance Z_T, and R_R and X_R are the corresponding components of the receiver impedance Z_R.

The power W expended in the receiver—which it is desired to make a maximum—is, by definition of the term *effective resistance*, the square of the current multiplied by the effective resistance of the receiver, or

$$W = \left| \frac{E}{R_T + jX_T + R_R + jX_R} \right|^2 R_R = \frac{E^2 R_R}{(R_T + R_R)^2 + (X_T + X_R)^2} \tag{2}$$

If it is assumed that the direct current flowing through the transmitter is fixed by other considerations, the simplest way to improve the efficiency of such a circuit, as is shown in Fig. 1, is by changing the number of turns in the receiver winding. According to the assumptions previously discussed in Sect. 1.22 such a change will affect only the *modulus* (i.e absolute magnitude) of the impedance of the receiver and not its phase angle. In other words $Z_R \equiv R_R + jKR_R = R_R(1 + jK)$ where K is a constant but R_R or $|Z_R|$* is a variable. Putting the value of KR_R for X_R in equation (2), we get

$$W = \frac{E^2R_R}{(R_T + R_R)^2 + (X_T + KR_R)^2} \tag{3}$$

Differentiating this expression with respect to R_R, it is found that the power W in the receiver is a maximum when

$$\sqrt{R_T^2 + X_T^2} = R_R\sqrt{1 + K^2} \qquad \text{or} \qquad |Z_T| = |Z_R| \tag{4}$$

In other words the absolute magnitude of the receiver impedance must be made equal to that of the transmitter. Under these conditions, the power in the receiver becomes

$$W_{\text{max.}} = \frac{E^2}{2[R_T + KX_T + \sqrt{(R_T^2 + X_T^2)(1 + K^2)}]} \tag{5}$$

Suppose now that the receiver impedance is variable in phase as well as in magnitude. Such an effect can evidently be accomplished, for any given frequency, by first winding the receiver to the proper absolute impedance, and by then inserting a reactance, such as a condenser or inductance, of suitable value, in series with the receiver until the desired phase angle is obtained.

In other words, assume that *both* the effective resistance R_R and the effective reactance X_R of the receiver are variable and that the problem is, as before, to find what impedance the receiver should have in order that it may absorb the maximum possible power from the transmitter.

If, in equation (2), W is differentiated with respect to R_R, the power W is found to be a maximum when

$$R_R^2 = R_T^2 + (X_T + X_R)^2 \tag{6}$$

Similarly, differentiating with respect to X_R, it is found that the power W is a maximum when

$$X_R = -X_T \tag{7}$$

Solving equations (6) and (7) simultaneously, the power W in the receiver is found to be an absolute maximum when both

$$R_R = R_T \qquad \text{and} \qquad X_R = -X_T \tag{8}$$

Putting the values of (8) in equation (2) we get

$$W_{\text{max.}} = \frac{E^2}{4R_T} \tag{9}$$

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* Vertical bars are employed throughout to indicate the modulus or the absolute (scalar) magnitude of the complex quantity enclosed thereby.

*Therefore, the maximum amount of power which it is possible for a receiving device to absorb from a source having an electromotive force E, and an effective resistance R_T, is the square of the electromotive force (E^2) divided by four times the effective resistance ($4R_T$), and is obtained when the impedances of the source and the receiver are conjugates.**

By comparing (5) and (9) it is seen that it is practically always possible to draw more power into the receiver when both the effective resistance R_R and the reactance X_R of the receiver are variable, than it is when only the absolute magnitude of the receiver impedance is variable. If, however, the phase angle of the receiver impedance is the negative of that of the transmitter, equations (5) and (9) become identical, and the absolute maximum power is drawn from the transmitter in either case.

3.2 Simple Two-Way Transmission—Series Circuit.—In practically all commercial circuits it is, of course, necessary to provide means for communication in both directions. This evidently involves a transmitter and receiver at each end of the circuit. The simplest possible form of such a circuit is when the four instruments are all connected in series, as is shown in Fig. 2.

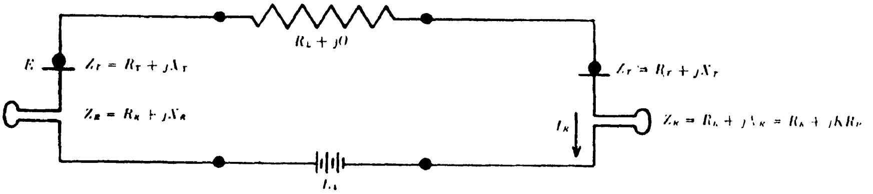

FIG. 2. Simple series circuit for two-way transmission.

Assume a circuit, such as that of Fig. 2, in which the line is short so that its shunt capacity is negligible but that the line has, nevertheless, a series resistance of R_L ohms which cannot be neglected. Assume, furthermore, that the transmitter impedance is pure resistance (as it would be in the case of a carbon button transmitter) or in other words that $X_T = 0$, that the phase angle of the receivers is constant—i.e., that the ratio K of the reactance of the receivers to that of their effective resistance is constant, and that the problem to be solved is to find the impedance to which the receivers should be wound in order that the maximum combined transmitting and receiving efficiency may be obtained in the above circuit.

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* One impedance is said to be the conjugate of another impedance when their effective resistances are equal and their reactances are equal in magnitude but are opposite in sign.

Assuming an a-c. electromotive force E acting in one transmitter, the current flowing through the distant receiver is evidently

$$I_R = \frac{E}{2(Z_T + Z_R) + R_L} = \frac{E}{2(R_T + R_R + jKR_R) + R_L} \qquad (10)$$

The a-c. power absorbed in the receiver will be

$$W = |I_R^2| R_R = \left| \frac{E^2}{[2(R_T + R_R + jKR_R) + R_L]^2} \right| R_R \qquad (11)$$

Since *the absolute magnitude of any complex quantity is equal to the square root of the sum of the squares of its real and imaginary components,* it is evident that

$$W = \frac{E^2 R_R}{(2R_T + 2R_R + R_L)^2 + (2KR_R)^2} \qquad (12)$$

Assuming R_R to be the variable, it is found, by differentiation of the above expression, that the received power W is a maximum when

$$R_R\sqrt{1 + K^2} = R_T + \frac{R_L}{2} \quad \text{or} \quad |Z_R| = R_T + \frac{R_L}{2} \qquad (13)$$

This equation states as the condition for maximum voice-frequency efficiency that the absolute value of the impedance of each receiver should be made equal to the resistance of either transmitter plus one-half the resistance of the line. For example, if the value of the effective resistance R_T of the transmitter is taken as 40 ohms and if the line resistance R_L is assumed to be 100 ohms, the impedance of each receiver should be $40 + 100/2 = 90$ ohms absolute value—irrespective of the phase angle of the receiver. In the above equation it should, of course, be kept in mind that R_T is simply the effective resistance of the transmitter and that in some cases, as stated in Sect. 1.21, this may not be more than three-fourths of its d-c. resistance.

It is evident from the above that, if the line resistance R_L were zero, the absolute impedance of each receiver $|Z_R|$ should be made equal to the resistance R_T of either transmitter. On the other hand, if the resistance R_L of the line is large, as compared to the resistance R_T of the transmitters, then the absolute value of the receiver impedance $|Z_R|$ should be approximately equal to one-half the resistance of the line.

By combining equations (12) and (13)

$$W_{\text{max.}} = \frac{E^2}{4(2R_T + R_L)(1 + \sqrt{1 + K^2})} \qquad (14)$$

which is the power obtained in each receiver provided the receivers are wound to such an impedance as will give maximum circuit efficiency.

The square root of the ratio of equations (12) and (14) is

$$r = \sqrt{\frac{4R_R(2R_T + R_L)(1 + \sqrt{1 + K^2})}{(2R_T + 2R_R + R_L)^2 + (2KR_R)^2}} \qquad (15)$$

This expression gives the value of the current ratio (I_2/I_1) corresponding to the number of attenuation units, miles of standard cable, or TU, by which the efficiency of the circuit indicated in Fig. 2 could be improved, provided the receiver impedance is given the best possible value—as indicated by equation (13). The conversion of this current ratio into any of the various transmission units may be made by means of the formulae given in Chapter II.

As an example of the use of the above formula consider a circuit having the following constants at 800 cycles:

$$\begin{array}{llll} R_T = 40 & & X_T = 0 \\ R_L = 100 & & X_L = 0 \\ R_R = 20 & & K = 1.73 \end{array}$$

From the assumed values of R_R and K the receiver impedance is found to be $40\underline{/60^\circ}$ ohms or, in other words, equal in absolute value to the transmitter resistance R_T. It has already been found, however, from (13) that, with a transmitter resistance of 40 ohms and with a line resistance of 100 ohms, the circuit would have the greatest possible voice-frequency efficiency with a receiver impedance of $90\underline{/60^\circ}$ ohms.

In order, then, to see how much voice-frequency loss has been incurred by winding the receiver to an impedance of $40\underline{/60^\circ}$ ohms, instead of to the ideal value of $90\underline{/60^\circ}$ ohms, it is only necessary to put the assumed data in equation (15) and find the corresponding value of r. This value of r is found to be .901, which corresponds to .104 napier, to .96 mile of standard cable or to .91 TU. Consequently, by winding the receiver with finer wire—keeping the spools full—to an impedance of $90\underline{/60^\circ}$ ohms, the voice-frequency transmission efficiency of the circuit would be improved by approximately one mile of standard cable.

In order to get the actual difference in transmission efficiency under the above conditions it would, in this particular case, be necessary to consider the difference in the d-c. or *current supply losses* for the two conditions. By increasing the impedance of the receiver, as indicated, the a-c. efficiency of the circuit would be increased by .104 napier but, at the same time, the increased d-c. resistance of the receivers and the consequent reduction in the amount of direct current flowing through the transmitters—provided the latter were supplied, as is indicated in Fig. 2, by a battery of voltage E_1 in series with the instruments—would reduce the amount of a-c. power given out by the transmitters. In a similar way, if electromagnet receivers were employed, the reduced amount of direct current flowing through them might also cut down their *inherent efficiency,* that is, their ability to convert electrical power into sound power. Consequently, in any such circuit it is always necessary to consider the algebraic sum of the a-c. and the d-c. losses in order to determine when the maximum transmission efficiency of the circuit has been obtained.

It has been shown that (12) gives the received power in any simple series circuit containing two transmitters and two receivers—the constants of which are known—while (14) gives the received power in such a circuit, provided the receivers are wound to the best possible impedance for maximum overall a-c. efficiency. In general, however, equation (14) does not give the maximum a-c. efficiency that can be obtained.

As has been pointed out in the case of the simple series circuit, in which the transmitter was directly connected to the receiver, the absolute maximum efficiency of such a circuit requires that the reactance of the receiver be annulled ($X_R = - X_T$). This can be done, of course, at any frequency by simply inserting a condenser of the proper capacity in series with the receiver—on the assumption that the difficulties due to battery supply are to be neglected. It also requires that the effective resistance of the receiver be made equal to that of the transmitter and, finally, that the effective resistances of the transmitters and receivers be made so large that the series resistance introduced by the line is negligible in comparison with them. In such a case K and R_L in equation (14) are zero and the received power is

$$W_{\text{max.}} = \frac{E^2}{16R_T} \tag{16}$$

By comparing (16) with (9) it is seen that the power received, in the case where both a transmitter and a receiver are used at each end of the circuit, is only one-fourth of that obtained in the receiver in the case where there was simply one transmitter operating directly into a receiver. Consequently, it may be stated as a general proposition that *in any ideal invariable circuit* in which two-way transmission is required, the power received will be only one-fourth of that which would exist in a similar circuit in which*

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* A circuit is said to be invariable when it is electrically identical in both its receiving and transmitting conditions. See Chapter X.

only one instrument is used at each end of the circuit. This power ratio of one-fourth corresponds to .693 *napier or to approximately* 6.4 *miles of standard cable.*

3.3 Simple Two-Way Transmission—Parallel Circuit.—Although the series type of circuit is, so far as is known, the only type of local or *interphone* circuit in extensive use, there are, under certain conditions, material advantages, from a transmission standpoint, in using a parallel type of circuit, such as is shown in Fig. 3. It takes only very rough computations of the efficiency of the ordinary series type of circuit to show conclusively that by far the largest losses in such circuits are those which have been called battery supply losses. Even with the longest interphone circuits (approximately one-half mile of No. 22 A.W.G. wire) the voice-frequency loss is, in general, less than one mile. Consequently, it is evident that any great improvement in the transmission efficiency of such circuits must lie in the direction of improved current supply conditions. Let us first, then, make a comparison of the series type of circuit with the parallel circuit from a current supply or *direct current point of view.*

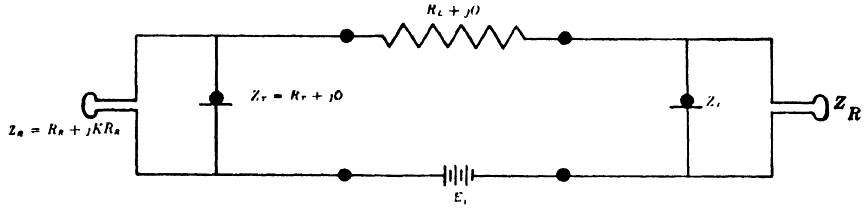

FIG. 3. Simple parallel circuit for two-way transmission.

Assume in each case a battery of voltage E_1 and that the d-c. resistances of the receiver, transmitter and line are, respectively, R_R, R_T and R_L. The direct current in the transmitters I_{T_S} or in the receivers I_{R_S} in the series circuit (Fig. 2) is then

$$I_{T_S} = I_{R_S} = \frac{E_1}{2(R_R + R_T) + R_L} \tag{17}$$

In the parallel type of circuit the direct current I_{TP} in the transmitters is

$$I_{T_P} = \frac{E_1 R_R}{R_L(R_R + R_T) + 2R_R R_T} \tag{18}$$

and the direct current I_{R_P} in the receivers is

$$I_{R_P} = \frac{E_1 R_T}{R_L(R_R + R_T) + 2R_R R_T} \tag{19}$$

Taking the ratio of (18) to (17), it is found that the ratio r_T of the direct current in the transmitter in the parallel circuit to that flowing in the transmitter of the series circuit is

$$r_T = \frac{2R_R(R_R + R_T) + R_R R_L}{R_L(R_R + R_T) + 2R_R R_T} \tag{20}$$

If the line resistance R_L is so small that it can be neglected, equation (20) gives

$$r_T = \frac{R_R + R_T}{R_T} = 1 + \frac{R_R}{R_T} \tag{21}$$

If the line resistance R_L is equal to $2R_R^2/R_T$, equation (20) gives

$$r_T = 1 \tag{22}$$

Hence, for all lines in which the line resistance is smaller than $2R_R^2/R_T$ there will be more direct current in the transmitters of the parallel circuit than in those of the series circuit and, hence, the parallel circuit will, for those line conditions, be the more efficient from a current supply standpoint.

For the sake of a specific example, consider the circuits when using transmitters the d-c. resistance R_T of each of which is equal to 40 ohms and receivers the resistance R_R of each of which is equal to 83 ohms. With these assumptions it is found from the relation $R_L = 2R_R^2/R_T$ that the transmitters in the parallel type of circuit would receive more direct current than they would in the series type of circuit—provided the line resistance R_L did not exceed 345 ohms, a resistance seldom approached in so-called interphone or local telephone circuits. Since, as previously stated, the current supply losses are by far the most serious in such circuits, the use of a parallel type of circuit for such work would appear to be highly advantageous, unless a very low resistance receiver (such as one of the electromagnet type) and a very high resistance transmitter were used, in which case it is probable that the line resistance R_L would be greater than $2R_R^2/R_T$ and the series circuit would be preferable to the parallel circuit.

By taking the ratio of I_R in equation (19) to that in equation (17), it is found that there will be more direct current in the receivers of the parallel circuit than in those of the series circuit if $R_L < 2R_T^2/R_R$. In the case just considered where $R_T = 40$ ohms and $R_R = 83$ ohms, the receivers would get more direct current in those cases in which the line resistance did not exceed about 40 ohms. On the other hand, if transmitters of 500 ohms resistance and receivers of 80 ohms resistance were used, there would be more direct current in the receivers in the parallel circuit case for all line resistances up to 6,250 ohms. Hence, the parallel type of circuit would appear to have a marked advantage under such conditions.

Considering now the parallel type of circuit from a *voice-frequency point of view* and assuming an a-c. electromotive force E in one transmitter, the a-c. current in the distant receiver can be easily shown to be

$$I_R = \frac{EZ_TZ_R}{[Z_T + Z_R][Z_L(Z_T + Z_R) + 2Z_TZ_R]} \tag{23}$$

or

$$I_R = \frac{ER_T(R_R+jKR_R)}{(R_T+R_R+jKR_R)[R_L(R_T+R_R+jKR_R)+2R_T(R_R+jKR_R)]} \tag{24}$$

in which Z_T, Z_R, R_R, R_L and R_T now represent the a-c. or voice-frequency constants of the various circuit elements.

Whence the power in the distant receiver is

$$W = \frac{E^2R_T^2R_R^3(1 + K^2)}{[R_R^2(R_L + 2R_T)(1 - K^2) + R_T(R_LR_T + 2R_R\{R_L + R_T\})]^2 + 4[R_R^2K(R_L + 2R_T) + R_RKR_T(R_L + R_T)]^2} \tag{25}$$

By taking the ratio of the power W as given by (25) to that given by (12) a power ratio is obtained which gives a measure of the relative a-c. efficiencies of any simple parallel and series types of circuits.

If we assume a negligible line resistance (i.e., $R_L = 0$), equation (25) gives

$$W = \frac{E^2R_R}{4[R_R^2(1 + K^2) + R_T(R_T + 2R_R)]} \tag{26}$$

By differentiation, it can be shown that the power as given by (26) is a maximum when

$$R_T = R_R\sqrt{1 + K^2} = |Z_R| \tag{27}$$

In other words, with a line of negligible resistance the parallel circuit, as well as the series circuit, is most efficient from an a-c. standpoint when the absolute impedance of the receiver is equal to that of the transmitter.

CHAPTER IV

SIMPLE SERIES CIRCUIT—ON LONG CONNECTIONS

4.0 In all circuits previously considered, we have assumed that the telephone line was *electrically short;* that is, that the impedance of the receiving apparatus was the controlling factor in determining the amount of alternating current which flowed from the transmitting set. In an extremely large percentage of all commercial connections and especially in all toll or long distance connections, where it is extremely important to obtain maximum efficiency, the impedance of the listening or distant set has a negligible effect upon the amount of current or power leaving the transmitting or near set. The reason for this can be seen by studying the formula * for the impedance of an *electrically long* line, i.e., one having a large attenuation constant. Consequently, in the commercial case we have a given line impedance Z_L to which the subscriber's set is connected—this impedance being independent, for all practical purposes, of the constants of the distant subscriber's set. The problem, in the simple series case—without any induction coil, etc.—can then be easily attacked by considering such a circuit as is shown in Fig. 1.

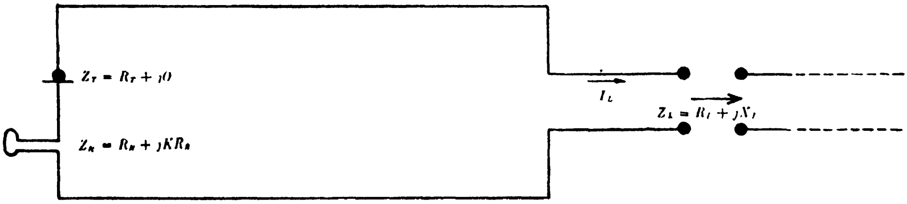

FIG. 1. Simple series circuit connected to an electrically long line of impedance Z_L.

In an ideal series circuit of this type—that is, one in which the impedance of the instruments can be varied as desired to give maximum combined transmitting and receiving—or overall efficiency—it has already been proven that the effective resistance of the transmitter R_T would be equal to that of the receiver R_R and that the sum of R_T and R_R would be equal to the effective resistance of the line R_L. Furthermore, the reactance of the line would be exactly annulled by that of the receiver, or

$$X_L = -KR_R$$

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* See equation (48) in Sect. 11.6.

In general, however, we do not have commercially available an efficient transmitter whose effective resistance R_T is as large as $R_L/2$; moreover, it is not usually practicable to vary the reactance of the receiver KR_R without at the same time varying its effective resistance R_R. The actual problem ordinarily confronting us is, having given as fixed the values R_L, X_L, R_T and K, to determine for what value of R_R the series circuit shown in Fig. 1 is most efficient.

4.1 Transmitting Efficiency.—If we first consider simply the transmitting efficiency of the circuit, and assume an a-c. electromotive force E acting in the transmitter, the current entering the line is

$$I_L = \frac{E}{(R_R + R_L + R_T) + j(X_L + KR_R)} \tag{1}$$

and the power W_L delivered to the line is

$$W_L = \frac{E^2 R_L}{(R_R + R_L + R_T)^2 + (X_L + KR_R)^2} \tag{2}$$

With R_R as a variable it is easy to show from (2) that maximum power is delivered to the line, and hence, that the set has maximum transmitting efficiency when

$$R_R = \frac{-(R_L + R_T + X_L K)}{1 + K^2} \tag{3}$$

Since a negative value of the effective resistance R_R is something physically unattainable, (3) means that, if X_L is positive, maximum transmitting efficiency is obtained when $R_R = 0$, this being the nearest physical value to a negative quantity that R_R can have. If on the other hand X_L is negative and is of such a magnitude that $X_L K$ is greater—in absolute value—than $(R_L + R_T)$, then the value of R_R as given by (3) is positive and is physically attainable. Consequently, under these last conditions, (3) gives us the value of R_R which will make the set have maximum transmitting efficiency. Under all other conditions the receiver must be omitted ($R_R = 0$) if the set is to have maximum transmitting efficiency.

4.2 Receiving Efficiency.—Assume an electromotive force e acting in series with the line impedance $R_L + jX_L$. The justification for this assumption is given in Chapter VIII. The current in the receiver is then

$$I_R = \frac{e}{(R_R + R_L + R_T) + j(X_L + KR_R)} \tag{4}$$

and the power dissipated in the receiver is

$$W_R = \frac{e^2 R_R}{(R_R + R_L + R_T)^2 + (X_L + KR_R)^2} \tag{5}$$

Assuming as before that R_R is the variable, it can be shown that the power W_R in the receiver is a maximum when

$$|Z_R| = \sqrt{R_R^2(1 + K^2)} = \sqrt{(R_L + R_T)^2 + X_L^2} = |Z_L + Z_T| \tag{6}$$

In other words, *maximum receiving efficiency is obtained when the absolute value of the impedance of the receiver is made equal to that of the impedance to which the receiver terminals are connected.*

4.3 Combined Transmitting and Receiving Efficiency.—The total combined transmitting and receiving efficiency of the set will be a maximum when the product of W_L and W_R, as given by (2) and (5), is a maximum. This product is

$$W_L W_R = \frac{E^2 e^2 R_R R_L}{[(R_R + R_L + R_T)^2 + (X_L + KR_R)^2]^2} \tag{7}$$

Assuming R_R to be the variable, we find from (7) that maximum overall or combined transmitting and receiving efficiency of the set is obtained when

$$R_R = \frac{\sqrt{(R_L + R_T + X_L K)^2 + 3(1 + K^2)[(R_L + R_T)^2 + X_L^2]}}{3(1 + K^2)} - \frac{R_L + R_T + X_L K}{3(1 + K^2)} \tag{8}$$

From (6) and (8) it can be shown, that for typical trunk conditions, the impedance to which the receiver of a simple series set should be wound in order that the set may have maximum overall or combined transmitting and receiving efficiency is not, in general, sufficiently high to make the set impedance as high as that of the line impedance.

From (1) and the corresponding value for the line current in an ideal invariable substation circuit (equation (21), Chapter X), we get

$$L_T = 21.13 \log_{10} \frac{\sqrt{(R_R + R_L + R_T)^2 + (X_L + KR_R)^2}}{2\sqrt{2R_L R_T}} \tag{9}$$

where L_T is the number of miles of standard cable by which the transmitting efficiency of any simple series circuit, of the type shown, is below that of an ideal invariable substation circuit.

In a similar way, we have when receiving

$$L_R = 21.13 \log_{10} \frac{\sqrt{(R_R + R_L + R_T)^2 + (X_L + KX_L)^2}}{2\sqrt{2R_R R_T}} \tag{10}$$

where L_R is the number of miles (computed for $\omega = 5000$) by which the receiving efficiency of any simple series circuit is below that of an ideal invariable substation circuit.

If the reactance X_L of the line is zero, (8) reduces to

$$R_R = \frac{(R_L + R_T)[\sqrt{4 + 3K^2} - 1]}{3(1 + K^2)} \tag{11}$$

The product of equations (1) and (4) gives

$$I_L \times I_R = \frac{Ee}{[(R_R + R_L + R_T) + j(X_L + KR_R)]^2} \tag{12}$$

Considering simply the absolute value of (12) we have

$$|I_L \times I_R| = \frac{Ee}{(R_R + R_L + R_T)^2 + (X_L + KR_R)^2} \tag{13}$$

which is, as previously stated, a criterion of the total or overall efficiency of the circuit. The corresponding criterion of the combined transmitting and receiving efficiency of the *ideal circuit* is, as will be proved later (equation (23), Chapter X),

$$|I_L \times I_R| = \frac{Ee}{8R_L\sqrt{R_R R_T}} \tag{14}$$

The ratio r of (13) to (14) is

$$r = \frac{8R_L\sqrt{R_R R_T}}{(R_R + R_L + R_T)^2 + (X_L + KR_R)^2} \tag{15}$$

and is the current ratio corresponding to the number of miles which the

TABLE I

Given				Impedance of receiver for max. overall efficiency (Eq. (8))				Total a-c. efficiency of set—miles below ideal (from Eq. 15)
R_L	X_L	R_T	K	R_R	KR_R	$\|Z_R\|$	θ	Loss
700	0	40	0	247	0	247	$\underline{/0^\circ}$	5.1
700	0	40	1.732	160	277	320	$\underline{/60^\circ}$	6.3
700	0	40	4.33	84	366	375	$\underline{/77^\circ}$	8.4
700	0	40	∞	0	428	428	$\underline{/90^\circ}$	∞
700	− 200	40	0	260	0	260	$\underline{/0^\circ}$	5.5
700	− 200	40	1.732	191	331	382	$\underline{/60^\circ}$	5.4
700	− 200	40	4.33	102	440	452	$\underline{/77^\circ}$	7.0
700	− 200	40	∞	0	514	514	$\underline{/90^\circ}$	∞

actual circuit is below, in total transmitting and receiving efficiency, that of the ideal invariable circuit.

In order to get a clearer idea of the effect of the reactance of the line impedance, there are given in Table I a few numerical cases, computed at a frequency of 800 cycles.

4.4 Summary.

4.41 Best Impedance for Receiver.—From an inspection of the data in the above table it is evident that, for maximum combined transmitting and receiving efficiency of a simple series circuit and with the constants of apparatus ordinarily employed, the impedance of the receiver—and consequently roughly the impedance of the set—may lie anywhere between approximately .3 and .6 that of the line impedance. If, for example, the line and receiver impedances are both pure resistances, maximum efficiency is obtained when $R_R = (R_L + R_T)/3$ while, if the line impedance is pure resistance and the receiver impedance is pure reactance, the best impedance for the receiver is given by the relation $R_R = (R_L + R_T)/\sqrt{3}$. In general, the larger the negative reactance of the line or the larger the phase angle of the receiver, the greater will be the receiver impedance. Since in formula (8) R_T occurs only in connection with the effective resistance of the line R_L and since in most cases encountered commercially the transmitter resistance R_T is negligibly small in comparison with that of the line R_L, it is evident that the best value for the receiver impedance will not change appreciably with moderate changes in the transmitter resistance.

Moreover, on the assumption previously made, namely that it is possible to vary only the *absolute* value of the impedance of the receiver and not its phase angle, we have shown that the circuit will have maximum transmitting efficiency when (3) is fulfilled, maximum receiving efficiency when (6) is fulfilled, and maximum overall or combined transmitting and receiving efficiency when (8) is satisfied.

4.42 A-C. Efficiency of Simple Series Circuit.—With a 40 ohm transmitter and a receiver having a phase angle of from 45° to 60° and an absolute impedance such as to give the circuit maximum overall efficiency, the transmitting efficiency of a simple series circuit, when connected to a typical telephone circuit, is approximately seven miles below that of an ideal invariable substation circuit. Under the same assumptions as made above, the receiving efficiency of a simple series circuit will be about two miles better than that of the ideal invariable circuit. Consequently, the overall efficiency of such a typical circuit is about five miles below that of the ideal invariable circuit.

CHAPTER V

KIRCHHOFF'S LAWS AND THE RECIPROCITY THEOREM

5.0 In solving circuit problems involving either direct or alternating currents, use is practically always made of certain fundamental laws or principles—the most important of which are called Kirchhoff's Laws.

5.1 Kirchhoff's Laws.—These laws may be stated as follows:

(1) The algebraic sum of all the currents flowing toward a junction point in any network of conductors is zero. It is to be noted that a current flowing *from* a point is the algebraic equivalent of a negative current flowing *toward* that point.

(2) Around any closed path in a network of conductors, the algebraic sum of the e.m.f.'s is equal to the algebraic sum of all the potential drops. Otherwise stated, the algebraic sum of all potentials around a closed path or mesh is zero.

5.11 Simple Two-Mesh Circuit.—The method of employing these laws in the solution of circuit problems may perhaps be best understood by considering a simple case such as that shown in Fig. 1. In this circuit, in which the voltages and currents may be either the d-c. values, the a-c. instantaneous, or r.m.s. values, the voltages and currents are assigned arbitrary directions. When the directions of the currents have been chosen, the first of Kirchhoff's Laws should then be applied. For example, at the point P in the above circuit it has been arbitrarily assumed that the current I_A is flowing *into* the junction while the currents I_B and I_C are flowing *away* from the junction. We then have

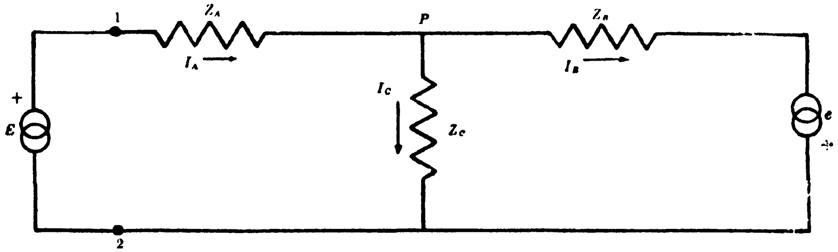

FIG. 1. Two-mesh circuit involving three impedances and two e.m.f.'s.

$$I_A = I_B + I_C \tag{1}$$

It is then necessary to determine the values of simply two currents, I_B

and I_C. This is, moreover, a general condition, namely, that after applying the first law there will always be as many independent variables as there are distinct meshes or loops in the circuit under consideration.

Considering next the left hand mesh or loop in Fig. 1, and applying Kirchhoff's second law, we get

$$E = I_A Z_A + I_C Z_C = (I_B + I_C) Z_A + I_C Z_C \tag{2}$$

Similarly, by considering the right hand mesh or loop, and applying Kirchhoff's second law, we have

$$e = I_B Z_B - I_C Z_C \tag{3}$$

The negative sign before the last term is required in view of the fact that the current I_C was originally drawn as *opposing* the current I_B in the mesh under consideration. Solving these last two equations simultaneously we get

$$I_B = \frac{E Z_C + e(Z_A + Z_C)}{Z_A Z_B + Z_A Z_C + Z_B Z_C} \tag{4}$$

$$I_C = \frac{E Z_B - e Z_A}{Z_A Z_B + Z_A Z_C + Z_B Z_C} \tag{5}$$

$$I_A = I_B + I_C = \frac{E(Z_B + Z_C) + e Z_C}{Z_A Z_B + Z_A Z_C + Z_B Z_C} \tag{6}$$

If, for an example, we assume the following impedances and voltage values:

$$\begin{aligned} Z_A &= 100\,\underline{/60^\circ} \\ Z_B &= 200\,\underline{/0^\circ} \\ Z_C &= 50\,\underline{/45^\circ} \\ E &= 100\,\underline{/0^\circ} \\ e &= 10\,\underline{/0^\circ} \end{aligned}$$

we find that

$$I_A = .73\,\overline{/52^\circ\, 23'}, \qquad I_B = .195\,\overline{/14^\circ\, 22'}, \qquad I_C = .588\,\overline{/64^\circ\, 9'}$$

In solving such problems by means of Kirchhoff's Laws, it is usually easier to apply the first law while designating the currents in the various arms. In other words, in Fig. 1 instead of writing I_C for the current in the impedance Z_C we could have written immediately $(I_A - I_B)$. This procedure reduces the number of unknown currents to only two.

Referring again to Fig. 1, if we let the electromotive force e be zero, the circuit is as shown in Fig. 2 and equations (4), (5) and (6) become respectively

$$I_B = \frac{EZ_C}{Z_AZ_B + Z_AZ_C + Z_BZ_C} \tag{7}$$

$$I_C = \frac{EZ_B}{Z_AZ_B + Z_AZ_C + Z_BZ_C} \tag{8}$$

$$I_A = \frac{E(Z_B + Z_C)}{Z_AZ_B + Z_AZ_C + Z_BZ_C} \tag{9}$$

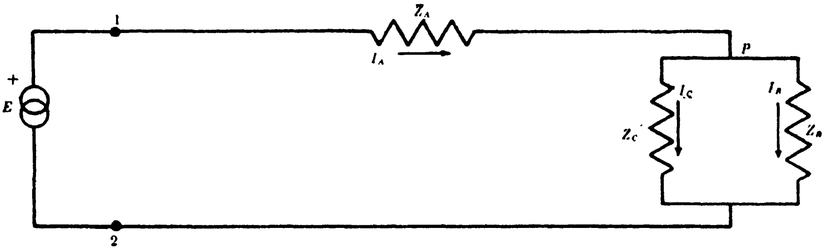

FIG. 2. Two-mesh circuit involving three impedances and one e.m.f.

From these last three equations certain well-known relations are almost immediately evident. For instance, the expression for I_A (see equation (9)) may also be written

$$I_A = \frac{E}{Z_A + \dfrac{Z_BZ_C}{Z_B + Z_C}} = \frac{E}{Z_R} \tag{10}$$

where Z_R is the impedance to the right of points 1–2 and consists of the impedance Z_A in series with an impedance $Z_BZ_C/(Z_B + Z_C)$. From this it is evident that *the effective impedance of two impedances in parallel is equal to the product of the two parallel impedances divided by their sum.* This is equivalent to saying that *the admittance of the combination is equal to the sum of the individual admittances.*

Two other important relations also may be obtained from the above equations. By dividing (7) by (9) we have

$$\frac{I_B}{I_A} = \frac{Z_c}{Z_B + Z_C} \quad \text{or} \quad I_B = I_A\left(\frac{Z_c}{Z_B + Z_C}\right) \tag{11}$$

Similarly, by dividing (8) by (9) we get

$$\frac{I_C}{I_A} = \frac{Z_B}{Z_B + Z_C} \quad \text{or} \quad I_C = I_A\left(\frac{Z_B}{Z_B + Z_C}\right) \tag{12}$$

From (11) and (12), it will be noted that the *ratio* of the current in either parallel arm to the external current I_A is entirely independent of the value of the external impedance Z_A. The relations expressed in these two equations will often be found very useful, especially in problems where only the *ratio* of the current at one point to that at another is desired.

5.12 Simple Bridge Circuit.—In order to illustrate more fully the use of Kirchhoff's Laws, we shall consider still another structure (see Fig. 3)

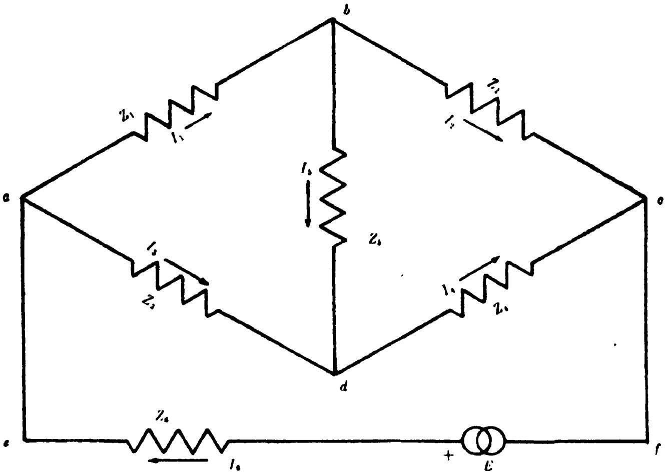

FIG. 3. Wheatstone bridge type of circuit.

and solve for the currents in the various arms. This type of three-mesh structure is the generalized form of the well-known Wheatstone bridge.

From Kirchhoff's first law we can write the following relations between the currents:

$$I_6 = I_1 + I_3 \tag{13}$$

$$I_6 = I_2 + I_4 \tag{14}$$

$$I_5 = I_1 - I_2 = I_4 - I_3 \tag{15}$$

From Kirchhoff's second law we have

$$E = I_6Z_6 + I_3Z_3 + I_4Z_4 \tag{16}$$

$$0 = I_1Z_1 + I_5Z_5 - I_3Z_3 \tag{17}$$

$$0 = I_2Z_2 - I_4Z_4 - I_5Z_5 \tag{18}$$

Equations (16), (17) and (18) are obtained by considering, respectively, the meshes *a d c f e*, *a b d* and *b c d*. We could, however, equally well have taken any other closed meshes, such as, for instance, the mesh *a b c f e* or the mesh *a b c d*. As far as the solution of the problem is concerned, it makes no difference what three independent meshes we choose to consider. In general, however, the solution of the simultaneous equations will be made somewhat less complicated by using the arm containing the electromotive force only once. Solving the above six equations simultaneously (see equations 21, 22 and 23 of Appendix A), we obtain the following expressions:

$$I_1 = \frac{E[Z_5(Z_3 + Z_4) + Z_3(Z_2 + Z_4)]}{H} \tag{19}$$

$$I_2 = \frac{E[Z_5(Z_3 + Z_4) + Z_4(Z_1 + Z_3)]}{H} \tag{20}$$

$$I_3 = \frac{E[Z_5(Z_1 + Z_2) + Z_1(Z_2 + Z_4)]}{H} \tag{21}$$

$$I_4 = \frac{E[Z_5(Z_1 + Z_2) + Z_2(Z_1 + Z_3)]}{H} \tag{22}$$

$$I_5 = \frac{E[Z_2Z_3 - Z_1Z_4]}{H} \tag{23}$$

$$I_6 = \frac{E[Z_5(Z_1 + Z_3) + (Z_2 + Z_4)(Z_1 + Z_3 + Z_5)]}{H} \tag{24}$$

in all of which

$$\begin{aligned} H \equiv (Z_1 + Z_2)(Z_3Z_4 + Z_5Z_6) &+ (Z_3 + Z_4)(Z_1Z_2 + Z_5Z_6) \\ &+ (Z_5 + Z_6)(Z_1Z_4 + Z_2Z_3) + Z_5(Z_1Z_3 + Z_2Z_4) \\ &+ Z_6(Z_1Z_2 + Z_3Z_4) \end{aligned} \tag{25}$$

It will be seen that from equation (23) we can immediately get the condition for zero current in the arm Z_5. This is the usual condition of balance in case the above circuit is used in an impedance bridge. It is obtained by equating the expression for I_5 to zero (assuming E and H to be finite). This gives

$$Z_2Z_3 = Z_1Z_4 \quad \text{or} \quad \frac{Z_1}{Z_2} = \frac{Z_3}{Z_4} \tag{26}$$

For reference purposes, there are given in Appendix E formulae, derived by Kirchhoff's Laws, for various types of commonly used circuits.

5.2 The Reciprocity Theorem.—In any system composed of *invariable bilateral* elements (i.e., in electrical circuits, where the resistance, inductance and capacity of each branch does not change with current saturation and where devices having *unilateral impedances*—like vacuum tubes or crystals—are not employed) it can be shown[*] that *if any electro-*

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[*] "Electricity and Magnetism," by C. Maxwell, Vol. I, p. 405.

"Theory of Variable Dynamical-Electrical Systems," by H. W. Nichols, *Physical Review*, 1917, Vol. 10, pp. 171–193.

"Electric Oscillations and Electric Waves," by G. W. Pierce, Chapter XIII.

motive force E is applied in any branch and the current I is measured in any other branch, their ratio (frequently called the transfer impedance) E/I is equal in magnitude and in phase to the ratio obtained if the positions of E and of I are interchanged. This theorem is known as the *reciprocity theorem.* It holds for practically all networks or circuit arrangements encountered in telephone practice.

As an example of this theorem and its use, consider the anti-side-tone substation circuit shown in Fig. 9 of Appendix E. It is seen from a comparison of formulae (63) and (70) in the appendix referred to that with an e.m.f. in Z_T, the expression for the current in Z_L (equation (63)) is exactly the same as it is provided the given e.m.f. is acting in Z_L and the current in Z_T is determined (equation (70)).

As an illustration of some of the uses which may be made of this principle consider an e.m.f. acting in Z_T and assume that the condition of anti-side-tone (i.e., no current in Z_R) be fulfilled or that $Z_L/Z_N \doteq \sqrt{Z_c/Z_D}$. Then the current in Z_L (or in Z_N) will be independent of the value of Z_R. Hence, by the reciprocity principle, with an e.m.f. acting in Z_L (or in Z_N) the current in Z_T will also be independent of the value of Z_R. Similarly, by assuming an e.m.f. acting in Z_R the current in Z_L (or in Z_N) will not be affected by the value of Z_T—provided, as before, the condition for no side-tone is fulfilled. Hence, by the reciprocity theorem, with an e.m.f. acting in Z_L (or Z_N) the current in Z_R will not be affected by the value of Z_T. This last statement means that *the receiving efficiency* (measured by the value of I_R with an e.m.f. acting in Z_L) *of any substation circuit that is actually anti-side-tone is not affected by changes in the resistance of the transmitter.* This is in marked contrast with other forms of substation circuits where effects on the receiving efficiency, due to variations in the transmitter resistance, are quite marked.

By similar methods of reasoning it can be shown that if the second *conjugacy condition* ordinarily existing in anti-side-tone or repeater circuits is fulfilled, namely, that an e.m.f. acting in Z_L (or Z_N) produces no current in Z_N (or Z_L)—this condition holding approximately when $Z_{CD}/Z_A \doteq Z_R/Z_T$—the value of the current I_L (or I_N) produced by an e.m.f. acting either in Z_T or Z_R will be independent of the value of Z_N (or Z_L).

From the preceding, it is seen that the reciprocity theorem is a very important principle, which when taken in conjunction with certain circuit conditions—such as the conjugacy conditions referred to above—frequently makes it possible to analyze circuit conditions much more clearly and readily than could otherwise be done.

CHAPTER VI

CHARACTERISTICS OF IDEAL TRANSFORMERS

6.0 If we have telephone instruments of fixed impedances and a circuit of known a-c. constants, a problem which is frequently of great practical importance to communication engineers is, how can these elements best be associated with a transformer so as to obtain the maximum transmitting and receiving efficiency. As we shall see later, under the heading *Thévenin's Theorem,* a similar problem arises when we are permitted to enter a telephone circuit at any point and are called upon to determine what improvement can be made in the telephone circuit by inserting some sort of apparatus at that point. In either of the above cases, a partial answer usually is to insert an induction coil or transformer in the circuit. Consequently, a problem of extreme interest in telephone transmission work is to consider what are the characteristics of ideal transformers and how do actual transformers compare with ideal transformers in efficiency.

Since a transformer may be defined as any structure with two or more windings between which there exists *mutual impedance,* it is important first to define what is meant by *mutual impedance.* As it is ordinarily used in telephone work we may define *the mutual impedance between one pair of terminals and a second pair of terminals as the vector ratio of the electromotive force produced between either pair of terminals, on open circuit, to the current flowing between the other pair of terminals.* In view of the fact that the mutual impedance is a vector ratio, it may have either of two signs depending upon the assumed directions of the input current and the output voltage. The sign of the mutual impedance is, in general, identified by its effect in increasing (aiding) or in decreasing (opposing) the vector impedance of the meshes in which it exists. Examples illustrating the method of handling mutual impedance in simple series or parallel meshes are given in Sects. 6.1 and 6.2.

6.1 Impedance of Two Windings in a Series Aiding or Opposing Connection.

—With the above definition of mutual impedance in mind let us consider the impedance of two windings in a *series aiding* connection; that is, *a series connection in which the flux produced by the current in one winding is aided or increased by the current flowing in the other winding.*

If we assume the notation shown in Fig. 1, in which there is a mutual impedance Z_M between two windings whose *self impedances* are Z_P and Z_S (the self impedance of any structure is the vector ratio of the applied e.m.f. to the resultant current entering the structure when all other accessible circuits associated with it are open), we have by Kirchhoff's Laws and by our definition of mutual impedance

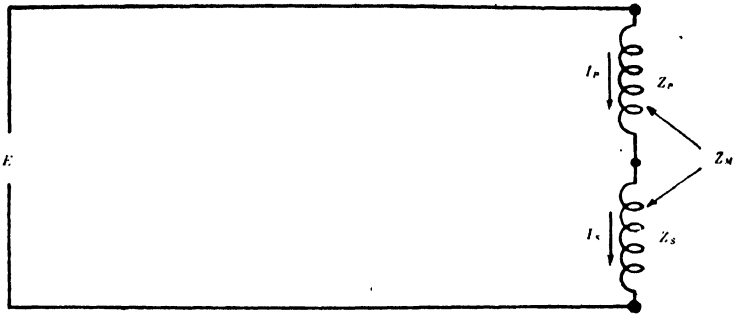

FIG. 1. Two windings of a transformer in a series aiding connection.

$$E = (I_PZ_P + I_SZ_M) + (I_SZ_S + I_PZ_M) \tag{1}$$

and

$$I_P = I_S \tag{2}$$

Combining (1) and (2), the impedance $Z_{\text{S.A.}}$ of the two windings in series aiding is

$$Z_{\text{S.A.}} \equiv \frac{E}{I_P} = \frac{E}{I_S} = Z_P + Z_S + 2Z_M \tag{3}$$

If we reverse the direction of either winding we will have what is commonly called a *series opposing* connection; i.e., *the flux produced by the current in one winding is opposed or decreased by the current flowing in the other winding.* The effect of changing the direction of any winding is evidently (from the definition of mutual impedance) to change the sign of any mutual impedance associated with that winding. In other words, if the direction of one of the windings is reversed so that we have a series opposing connection instead of a series aiding connection, the impedance $Z_{\text{S.O.}}$ is (from equation 3)

$$Z_{\text{S.O.}} = Z_P + Z_S - 2Z_M \tag{4}$$

In an *ideal transformer,* which is defined as *a transformer which neither stores nor dissipates energy,* there is no d-c. resistance, there are no eddy current or hysteresis losses, etc., and there is perfect flux linkage between the windings. In such a transformer it can be shown, by applying an e.m.f. to either winding alone and to the two windings in series aiding, that the self impedances Z_P and Z_S and the mutual impedance Z_M must all be either infinite or zero if there is to be no storage of energy or no energy dissipation. Then, if an ideal transformer is to deliver any current to a finite impedance Z_R, it can be proved from equation (3) in Sect. 7.1 that *the self impedances Z_P and Z_S must be infinitely great and Z_M must equal $\sqrt{Z_P Z_S}$. Hence, in any ideal transformer the self impedances Z_P and Z_S are infinitely large pure reactances and*

$$Z_M = \sqrt{Z_P Z_S} \tag{5}$$

These are two very important facts to keep in mind.

It is evident from (3) that the series aiding impedance $Z_{\mathrm{S.A.}}$ of a transformer in which equation (5) holds is

$$Z_{\mathrm{S.A.}} = Z_P + Z_S + 2\sqrt{Z_P Z_S} = (\sqrt{Z_P} + \sqrt{Z_S})^2 \tag{6}$$

or if the two windings are equal we have

$$Z_{\mathrm{S.A.}} = 4Z_P = 4Z_S \tag{7}$$

Similarly, the series opposing impedance $Z_{\mathrm{S.O.}}$ of such a transformer is (from equation 4)

$$Z_{\mathrm{S.O.}} = Z_P + Z_S - 2\sqrt{Z_P Z_S} = (\sqrt{Z_P} - \sqrt{Z_S})^2 \tag{8}$$

which, in the case of a unity ratio transformer (i.e., one in which $Z_P = Z_S$), becomes zero.

6.2 Impedance of Two Windings in a Parallel Aiding or Opposing Connection.—Let us now consider the case where the two windings are in a parallel aiding or opposing connection. The direction of winding in the *parallel aiding connection* is similar to that of the series aiding connection; that is, *it is such that the flux which is produced by the current in one winding is aided or increased by the current flowing in the other winding.*

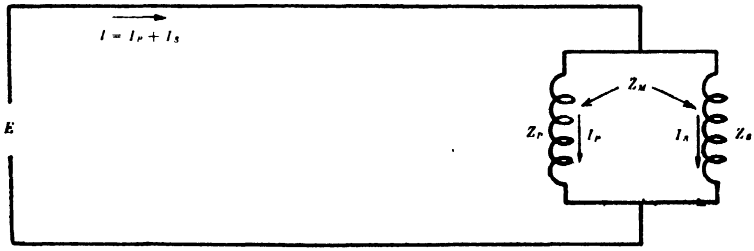

FIG. 2. Two windings of a transformer in a parallel aiding connection.

In such a circuit as is shown in Fig. 2, we have by Kirchhoff's Laws

$$E = I_P Z_P + I_S Z_M \tag{9}$$

and

$$I_P Z_P + I_S Z_M = I_S Z_S + I_P Z_M \tag{10}$$

Solving these equations simultaneously we get

$$I_P = \frac{E(Z_S - Z_M)}{Z_P Z_S - Z_M^2} \tag{11}$$

and

$$I_S = \frac{E(Z_P - Z_M)}{Z_P Z_S - Z_M^2} \tag{12}$$

or

$$I = I_P + I_S = \frac{E(Z_P + Z_S - 2Z_M)}{Z_P Z_S - Z_M^2} \tag{13}$$

Whence, the impedance $Z_{\text{P.A.}}$ of the two windings in parallel aiding is

$$Z_{\text{P.A.}} \equiv \frac{E}{I} = \frac{Z_P Z_S - Z_M^2}{Z_P + Z_S - 2Z_M} \tag{14}$$

As has been previously explained, we can immediately get from (14) the impedance $Z_{\text{P.O.}}$ of the two windings in parallel opposing connection by reversing the sign of Z_M. Whence, the parallel opposing impedance is

$$Z_{\text{P.O.}} = \frac{Z_P Z_S - Z_M^2}{Z_P + Z_S + 2Z_M} \tag{15}$$

If we assume, as before, an ideal transformer, that is, $Z_M = \sqrt{Z_P Z_S}$, it is apparent from (15) that the parallel opposing impedance is zero. On the other hand, if we put the above relation in the formula for the parallel aiding connection (equation (14)), we get zero except in the case where $Z_P = Z_S$, in which case the value is zero divided by zero or is indeterminate without further investigation.

As this is a condition which frequently comes up in telephone transmission problems, we will show how the actual value of the parallel aiding impedance may be determined. Assume that the transformer has a *coefficient of coupling* K, that is, assume that

$$Z_M \equiv K\sqrt{Z_P Z_S} = KZ_P = KZ_S \tag{16}$$

where K has a value that is less than unity. Then the parallel aiding impedance as given by (14) becomes

$$Z_{\text{P.A.}} = \frac{Z_P^2 - K^2 Z_P^2}{2Z_P - 2KZ_P} = \frac{1 + K}{2} Z_P \tag{17}$$

Whence, as the value of K approaches unity, we have

$$Z_{\text{P.A.}} \div Z_P = Z_S = Z_M \tag{18}$$

Hence, the parallel aiding impedance of such a transformer with a unity impedance ratio (that is, when $Z_P = Z_S$) is equal to the mutual impedance Z_M, whereas, if the impedance ratio is not unity—no matter how little Z_S may differ from Z_P—the impedance is zero.

6.3 Impedance Looking into Ideal Transformers.

6.31 Transformers with Two Separate Windings.—In the circuit shown in Fig. 3, if we assume that the arrows indicate the direction of the windings, that is, that 1–2–3–4 is a series aiding connection, we have by Kirchhoff's Laws

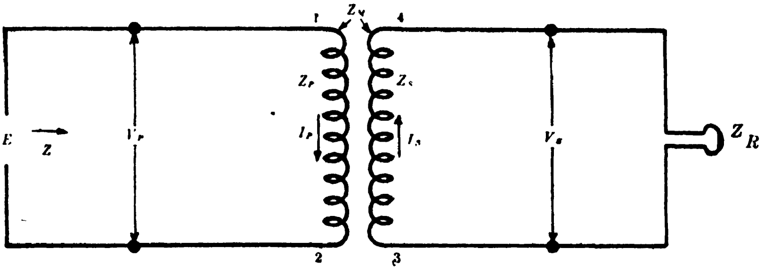

FIG. 3. Circuit in which a source of e.m.f. is directly connected to an absorber of power by a transformer having two separate windings.

$$E = I_P Z_P + I_S Z_M \tag{19}$$

and

$$I_S(Z_S + Z_R) + I_P Z_M = 0 \tag{20}$$

Whence

$$I_S = \frac{-EZ_M}{Z_P(Z_R + Z_S) - Z_M^2} \tag{21}$$

and

$$I_P = \frac{E(Z_R + Z_S)}{Z_P(Z_R + Z_S) - Z_M^2} \tag{22}$$

In the case of an ideal transformer the self impedances Z_P and Z_S are infinite, $Z_M = \sqrt{Z_P Z_S}$, and (21) and (22) become

$$-I_S = \frac{E\sqrt{Z_P Z_S}}{Z_P Z_R + (Z_P Z_S - Z_P Z_S)} = \frac{E\sqrt{Z_S}}{Z_R\sqrt{Z_P}} \tag{23}$$

and

$$I_P = \frac{EZ_S}{Z_P Z_R + (Z_P Z_S - Z_P Z_S)} = \frac{EZ_S}{Z_P Z_R} \tag{24}$$

Whence

$$-\frac{I_S}{I_P} = \sqrt{\frac{Z_P}{Z_S}} \tag{25}$$

The impedance, Z, looking into the 1–2 *terminals of the transformer is,* from (24),

$$Z \equiv \frac{E}{I_P} = \frac{Z_P}{Z_S} Z_R \tag{26}$$

or *it is the ratio of the self impedance Z_P of the adjacent winding to that of the distant winding multiplied by the impedance Z_R through which the distant winding is closed.* This is a very simple relation which is fulfilled fairly closely by most actual transformers and is a relation which should always be remembered.

The ratio of the voltages, V_P/V_S, across the two windings of the transformer is

$$\frac{V_P}{V_S} \equiv \frac{I_P Z}{I_S Z_R} = \frac{I_P}{I_S} \times \frac{Z}{Z_R} \tag{27}$$

From (25) and (26) it is evident that (27) reduces to

$$\frac{V_P}{V_S} = -\sqrt{\frac{Z_S}{Z_P}} \times \frac{\frac{Z_P}{Z_S} \times Z_R}{Z_R} = -\sqrt{\frac{Z_P}{Z_S}} \tag{28}$$

This relation, together with that given in (25), are both very simple ones and at the same time are ones that should always be kept in mind. Since *the number of turns* (N_P or N_S) *on any winding of an ideal transformer is proportional to the square root of the self impedance of the winding,* it is evident from (28) that *the ratio of the voltages across the windings of an ideal transformer is equal, numerically, to the ratio of the turns upon the two windings;* that is,

$$\left|\frac{V_P}{V_S}\right| = \frac{N_P}{N_S} \tag{29}$$

Also from (25) it is equally evident that *the ratio of the currents flowing in the windings of an ideal transformer is inversely proportional to the ratio of the numbers of turns on the windings* and, like the ratio of the voltages, is absolutely independent of the impedances or loads to which the windings are connected.

6.32 Auto-Transformers.—An auto-transformer, as ordinarily thought of, is a structure having two or more windings, at least one of which windings is common to more than one circuit or mesh. A simple two-winding auto-transformer is shown in Fig. 4. If we consider the windings to have the directions shown by the arrows (that is, 1–2–3–4 to be a series aiding connection), we get by Kirchhoff's Laws

$$E = I_S Z_S + (I_S - I_R) Z_M + I_R Z_R \tag{30}$$

and

$$(I_S - I_R)Z_P + I_S Z_M = I_R Z_R \tag{31}$$

Whence

$$I_S = \frac{E(Z_P + Z_R)}{(Z_P + Z_R)(Z_S + Z_M) + (Z_P + Z_M)(Z_R - Z_M)} \tag{32}$$

and

$$I_R = \frac{E(Z_P + Z_M)}{(Z_P + Z_R)(Z_S + Z_M) + (Z_P + Z_M)(Z_R - Z_M)} \tag{33}$$

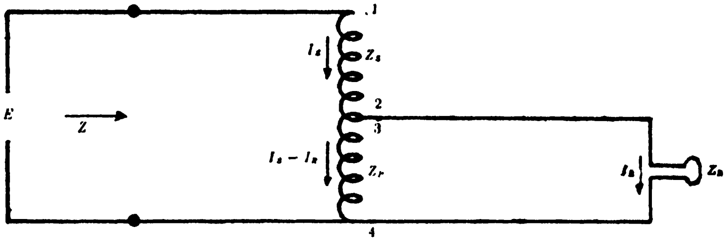

FIG. 4. Auto-transformer with mutual impedance, Z_M, between the windings.

From (32) the impedance looking into the *high* side (1–4 terminals) of the auto-transformer is

$$Z \equiv \frac{E}{I_S} = \frac{(Z_P + Z_R)(Z_S + Z_M) + (Z_P + Z_M)(Z_R - Z_M)}{Z_P + Z_R} \tag{34}$$

Assuming as before an ideal transformer—in which Z_P and Z_S are infinite and $Z_M = \sqrt{Z_P Z_S}$, we get from (34) that

$$Z = \frac{Z_P + Z_S + 2Z_M}{Z_P} Z_R = \left[\frac{\sqrt{Z_P} + \sqrt{Z_S}}{\sqrt{Z_P}}\right]^2 Z_R \tag{35}$$

Since the number of turns on any winding of an ideal transformer is proportional to the square root of the self impedance of the winding, (35) becomes

$$Z = \left[\frac{N_P + N_S}{N_P}\right]^2 Z_R \tag{36}$$

where N_P and N_S are the numbers of turns on the windings whose self impedances are Z_P and Z_S respectively.

It is evident from (35) that exactly as in the case of the transformer having two separate windings, *the impedance looking into the transformer is equal to the ratio of the self impedance of the adjacent winding* (*in this case the complete winding* 1–2–3–4) *to that of the distant winding* (*i.e., the winding linked with the load* Z_R *or winding* 3–4) *multiplied by the impedance* Z_R *through which the distant winding is connected.*

It should be pointed out that an auto-transformer as described above is always, for a given cost, somewhat more efficient than a similar transformer having two separate windings, and that the amount by which the auto-transformer is more efficient than the one equipped with separate windings depends upon the ratio of the self impedances of the windings of the transformers. That the former is the more efficient can easily be seen by considering the two types of transformers when the transformation ratio approaches unity. Under this condition the two windings of the separate winding transformer will each have essentially the same number of turns (and impedance), each winding taking up one-half of the total winding volume of the transformer. On the other hand, as the auto-transformer approaches the unity ratio condition, the auto-transformer approaches a one-winding device or a retardation coil bridged across the circuit. For the same high-frequency loss this one winding must have approximately the same inductance, or number of turns, as *each* of the two windings in the other type of transformer. Consequently—it can be given twice the winding volume or can be made to have roughly one-half of the d-c. resistance of each of the windings in the type of transformer which employs two separate windings.

6.4 Transformers Employing Three or More Windings.—Transformers employing three windings are used in various types of telephone circuits. The best known transformers of this sort are the so-called *hybrid transformers* used in "21" and "22" and "4-wire" repeater circuits for the purpose of making two branches of a circuit *conjugate* to each other; that is, such that an e.m.f. in one branch produces no current in another branch.

The formulae applying to such transformers may be derived in the usual way—as explained under Kirchhoff's Laws. For reference purposes rigorous formulae, applying to any transformer having three windings, are given in Appendix G. It is interesting to note that in the case of an *ideal* transformer having three separate windings whose self impedances are Z_1, Z_2 and Z_3, as shown in Fig. 5, the indicated impedance Z is:

$$Z = \frac{Z_3 Z_A Z_B}{Z_1 Z_B + Z_2 Z_A} \tag{37}$$

From (26) and (37) it can be deduced that *the impedance looking into any winding (called for convenience the primary winding) of an ideal transformer having N windings is equal to the self impedance Z_P of the primary winding multiplied by the product of all the load impedances (Z_A, Z_B, Z_C, Z_D, $\cdots$, Z_N) to which the other or secondary windings are connected, and divided by the sum of the products of the various self impedances of the* *secondary windings* (Z_1, Z_2, Z_3, Z_4, $\cdots$, Z_n) *and the various load impedances with which the different secondary windings are not directly associated.* For example, consider the ideal five-winding transformer shown in Fig. 6. The impedance Z looking into the Z_5 winding of this transformer is

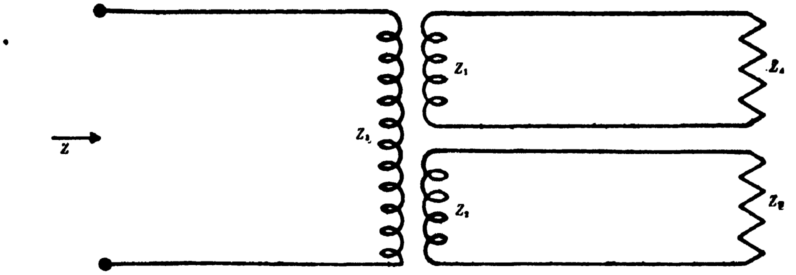

FIG. 5. Impedance looking into one winding of an ideal transformer having three separate windings.

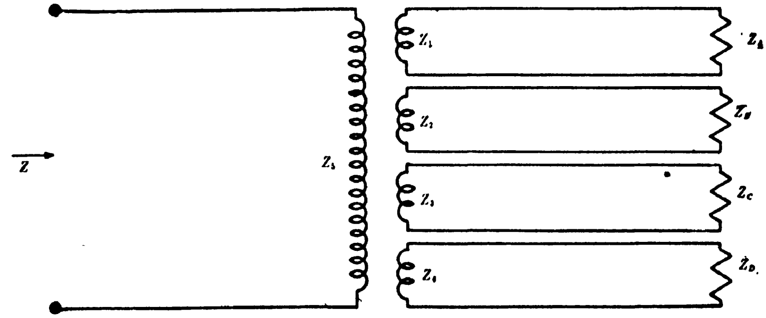

FIG. 6. Impedance looking into one winding of an ideal transformer having five separate windings.

$$Z = \frac{Z_5(Z_AZ_BZ_CZ_D)}{Z_1(Z_BZ_CZ_D) + Z_2(Z_AZ_CZ_D) + Z_3(Z_AZ_BZ_D) + Z_4(Z_AZ_BZ_C)} \qquad (38)$$

As an example, let us consider a specific case in which the values of Z_1, Z_2, Z_3, Z_4 and Z_5 are respectively in the ratio of 1 : 4 : 9 : 16 : 10 and the corresponding terminal impedances Z_A, Z_B, Z_C and Z_D are respectively 5 ohms, $10 + j10$ ohms, $0 - j50$ ohms, and $30 - j30$ ohms. With these data we find from the formulae given above that the impedance looking into the Z_5 winding is $14.1 \, /\overline{20°.3}$ ohms. Similar formulae for ideal auto-transformers can evidently be derived by application of the same general principle as enunciated above. Examples of such formulae for multi-winding auto-transformers are given in Appendix G.

CHAPTER VII

TRANSFORMER AND TRANSITION LOSSES

7.0 Three of the most important things a telephone transmission engineer has occasion to determine are: (1) how does the transmission efficiency of any actual transformer compare with that of the ideal transformer; (2) how does the actual transmission efficiency of any circuit compare with the efficiency which it would have if at some point in the circuit an ideal transformer were inserted; and (3) how does the actual transmission efficiency of any circuit compare with the efficiency which it would have if at some point in the circuit there were inserted the best possible *passive network*—that is, the best possible network that would not introduce power into the circuit, as would an amplifier. *The losses in transmission which would be eliminated by the insertion at any point of an ideal transformer having the optimum impedance ratio, or of the best possible passive network, are called respectively the transformer loss and the transition loss at that point.*

7.1 Derivation of Formulae.—In order to obtain a quantitative idea of the losses defined above, let us consider the circuit shown in Fig. 1.

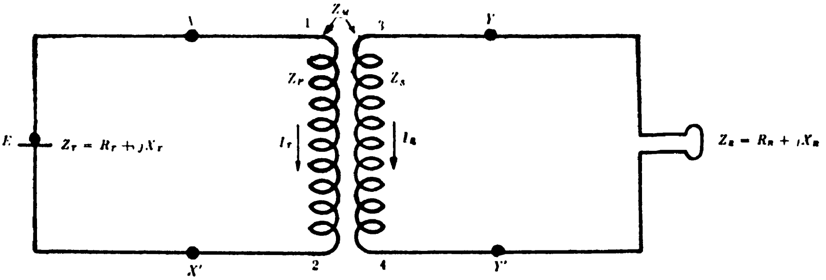

FIG. 1. General case in which a source of power is connected by a transformer having two separate windings to an absorber of power.

Assuming that 1–2–3–4 is the series aiding connection we have by Kirchhoff's Laws

$$E = I_T(Z_P + Z_T) + I_RZ_M \tag{1}$$

and

$$I_RZ_S + I_TZ_M + I_RZ_R = 0 \tag{2}$$

Solving these equations, we obtain for the current in Z_R

$$I_R = \frac{-EZ_M}{(Z_P + Z_T)(Z_R + Z_S) - Z_M^2} \tag{3}$$

The current which would have flowed in the receiver if the transformer had not been inserted is given by

$$I_R' = \frac{E}{Z_T + Z_R} \tag{4}$$

Equations (3) and (4) give at once a means of determining whether or not we have increased the current—and therefore the power—in Z_R by the insertion of the transformer.

Let us assume that the transformer which we have inserted is an *ideal transformer.* In such a transformer it has been shown that *the self impedance of each winding approaches infinity and* $Z_M = \sqrt{Z_P Z_S}$. Assume such a transformer, and let $r \equiv Z_S/Z_P$ or $Z_S = Z_P \times r$ and $Z_M \equiv \sqrt{Z_P Z_S} = Z_P \sqrt{r}$. Since Z_P is large compared with Z_T and Z_R, equation (3) becomes

$$\begin{aligned} I_R = \frac{-EZ_P\sqrt{r}}{Z_P(rZ_T + Z_R) + Z_T Z_R} &= \frac{-E\sqrt{r}}{rZ_T + Z_R} \\ &= \frac{-E\sqrt{r}}{r(R_T + jX_T) + (R_R + jX_R)} \end{aligned} \tag{5}$$

Since the absolute value of any complex quantity $X + jY$ is $\sqrt{X^2 + Y^2}$, the absolute value of I_R is

$$|I_R| = \frac{E\sqrt{r}}{\sqrt{(R_R + rR_T)^2 + (X_R + rX_T)^2}} \tag{6}$$

Differentiating this expression with respect to r we find that $|I_R|$ is a maximum when

$$r = \sqrt{\frac{R_R^2 + X_R^2}{R_T^2 + X_T^2}} = \frac{Z_R}{Z_T} \tag{7}$$

This expression is interesting in that it shows that *the ratio of the self impedances in an ideal transformer should, for maximum transmission efficiency, be equal to the ratio of the moduli or absolute (scalar) magnitudes of the impedances looking away in each direction from the point at which the transformer is to be inserted.*

If we let θ be the phase angle of Z_R (i.e., $Z_R = |Z_R|\underline{/\theta}$) and ϕ the phase angle of Z_T ($Z_T = |Z_T|\underline{/\phi}$), we get, by substituting the value of r, as given in equation (7), in equation (5):

$$
\begin{aligned}
|I_{R\,\mathrm{MAX.}}| &= \frac{E}{Z_R\sqrt{\left|\dfrac{Z_T}{Z_R}\right|}+Z_T\sqrt{\left|\dfrac{Z_R}{Z_T}\right|}} \\
&= \left|\frac{E}{\left[\,|Z_R|\cos\theta\sqrt{\left|\dfrac{Z_T}{Z_R}\right|}+|Z_T|\cos\phi\sqrt{\left|\dfrac{Z_R}{Z_T}\right|}\,\right]+j\left[\,|Z_R|\sin\theta\sqrt{\left|\dfrac{Z_T}{Z_R}\right|}+|Z_T|\sin\phi\sqrt{\left|\dfrac{Z_R}{Z_T}\right|}\,\right]}\right| \\
&= \frac{E}{|\sqrt{|Z_RZ_T|}[(\cos\theta+\cos\phi)+j(\sin\theta+\sin\phi)]|} \\
&= \frac{E}{\sqrt{|Z_RZ_T|}\,\sqrt{(\cos\theta+\cos\phi)^2+(\sin\theta+\sin\phi)^2}} \\
&= \frac{E}{\sqrt{2|Z_RZ_T|(1+\cos\theta\cos\phi+\sin\theta\sin\phi)}} \\
&= \frac{E}{\sqrt{2|Z_RZ_T|[1+\cos(\theta-\phi)]}} = \frac{E}{2\sqrt{|Z_RZ_T|}\cos\left(\dfrac{\theta-\phi}{2}\right)}
\end{aligned}
\tag{8}
$$

This expression gives us then the modulus or absolute magnitude of the maximum current that can be obtained in a receiving circuit of impedance Z_R (or $|Z_R|/\theta$) when connected through an ideal transformer having the best possible impedance ratio to a transmitting circuit of impedance Z_T (or $|Z_T|\underline{/\phi}$). If before inserting the ideal transformer we had annulled the reactance of the line in each direction by inserting a reactance of $-X_T$ ohms in series with Z_T and a reactance of $-X_R$ ohms in series with Z_R, (8) would become

$$|I_R'| = \frac{E}{2\sqrt{R_T \times R_R}} \tag{9}$$

and the power in Z_R would be

$$W_{\mathrm{MAX.}} = \frac{E^2}{4R_T} \tag{10}$$

As we have previously seen, however, this is the maximum possible power which can be drawn out of the source by any passive network. Consequently, *by first annulling the reactance of the line in each direction and by then inserting a perfect transformer of the best possible ratio, we obtain in the receiving impedance the absolute maximum amount of power which* *it is possible to draw from the source—the value of this power being the square of the e.m.f. (E^2) acting in the source divided by 4 times the effective resistance R_T of the source of the power.* A structure which, when inserted in a circuit, enables this maximum possible power to be absorbed in the receiving circuit is called an *ideal transducer.*

If we let $A_{\text{TRF.}}$ be the *transformer loss* (expressed in napiers), we get, by taking the ratio of the current in the receiver before and after inserting the transformer,

$$\epsilon^{-A_{\text{TRF.}}} = \frac{2\sqrt{|Z_R Z_T|}\cos\left(\frac{\theta-\phi}{2}\right)}{|Z_T+Z_R|} \tag{11}$$

The factor $2\sqrt{Z_R Z_T}/(Z_T+Z_R)$ has been commonly called the *reflection factor* of the impedances Z_T and Z_R. Hence, the *transformer factor* (11) is equal to the absolute magnitude of the reflection factor multiplied by the cosine of one-half of the angular difference between the two impedances. In other words, if we insert at any junction an ideal transformer of the best possible ratio, we will thereby obtain a *gain* that counteracts the *reflection loss* at that junction plus a *gain* that depends upon the angular difference of the impedances so connected.

If we let $r \equiv \left|\frac{Z_R}{Z_T}\right|$ and $\beta \equiv \theta - \phi$, then

$$\begin{aligned}\epsilon^{A_{\text{TRF.}}} &= \frac{|[|Z_T|\cos\phi + |Z_R|\cos\theta] + j[|Z_T|\sin\phi + |Z_R|\sin\theta]|}{2\sqrt{|Z_R Z_T|}\cos\frac{\beta}{2}} \\ &= \frac{\sqrt{|Z_T|^2 + |Z_R|^2 + 2|Z_R Z_T|\cos\beta}}{2\sqrt{|Z_R Z_T|}\cos\frac{\beta}{2}}\end{aligned} \tag{12}$$

Squaring and dividing by $|Z_T|^2$,

$$\epsilon^{2A_{\text{TRF.}}} = \frac{1 + 2r\cos\beta + r^2}{4r\cos^2\left(\frac{\beta}{2}\right)} = \frac{1 + 2r\cos\beta + r^2}{2r(1+\cos\beta)} \tag{13}$$

Whence

$$A_{\text{TRF.}} = 1/2\log_\epsilon\frac{1 + 2r\cos\beta + r^2}{2r(1+\cos\beta)} = 1.1513\log_{10}\frac{1 + 2r\cos\beta + r^2}{2r(1+\cos\beta)} \tag{14}$$

If we wish to express the *transformer loss,* $L_{\text{TRF.}}$, in miles of standard cable we have

$$L_{\text{TRF.}} = \frac{A_{\text{TRF.}}}{A_C} = \frac{1.1513}{A_C} \log_{10} \frac{1 + 2r\cos\beta + r^2}{2r(1 + \cos\beta)}$$
$$= \frac{298}{\sqrt{f}} \log_{10} \frac{1 + 2r\cos\beta + r^2}{2r(1 + \cos\beta)} \qquad (15)$$

in which f is the frequency employed and A_C is the attenuation constant of standard cable. If f is 796 cycles ($\omega = 5{,}000$) equation (15) gives

$$L_{\text{TRF.}} = 10.56 \log_{10} \frac{1 + 2r\cos\beta + r^2}{2r(1 + \cos\beta)} \qquad (16)$$

Curves plotted from this equation are given in Fig. 2 and are found to be very useful in determining whether or not the gain to be obtained by inserting a transformer at any point in a circuit is sufficiently large to be worth the expense involved. This gain is evidently another way of expressing the transformer loss $L_{\text{TRF.}}$.

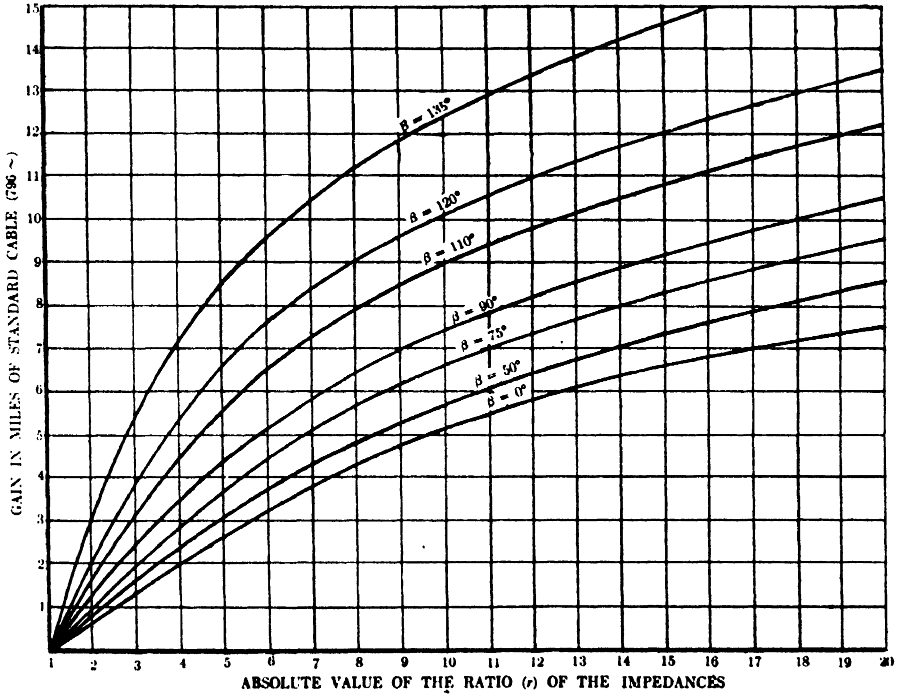

FIG. 2. Transformer loss curves—or curves showing the gain obtained by inserting an ideal transformer of the optimum ratio at the junction of two circuits whose impedances have the ratio $r \,/\, \pm\beta$. Computed for a frequency of 796 cycles.

If we let $A_{\text{TRT.}}$ be the transition loss and $A_O \equiv (A_{\text{TRT.}} - A_{\text{TRF.}})$, in which both $A_{\text{TRT.}}$ and A_O are expressed in attenuation units or napiers, we get from (8) and (9)

$$\epsilon^{-A_O} = \frac{2\sqrt{R_T \times R_R}}{\sqrt{2|Z_R Z_T|[1 + \cos(\theta - \phi)]}} = \frac{2\sqrt{|Z_R| \cos\theta \times |Z_T| \cos\phi}}{\sqrt{2|Z_R Z_T|[1 + \cos(\theta - \phi)]}} \tag{17}$$

or

$$\epsilon^{-A_O} = \frac{2\sqrt{|Z_T| r \cos\theta \times |Z_T| \cos\phi}}{\sqrt{2|Z_T|^2 r[1 + \cos(\theta - \phi)]}} = \sqrt{\frac{2\cos\theta\cos\phi}{1 + \cos(\theta - \phi)}} \tag{18}$$

Whence

$$A_O = 1/2 \log_\epsilon \frac{1 + \cos(\theta - \phi)}{2\cos\theta\cos\phi} = 1.1513 \log_{10} \frac{1 + \cos(\theta - \phi)}{2\cos\theta\cos\phi} \tag{19}$$

A_O has been called the *phase difference loss* since it is evident from (19) that its value depends solely upon the phase angles of the impedances and their difference.

If we let L_O be the difference between the transition loss and the transformer loss—or the phase difference loss—expressed in miles of standard cable, we have

$$L_O = \frac{A_O}{A_C} = \frac{1.1513}{A_C} \log_{10} \frac{1 + \cos(\theta - \phi)}{2\cos\theta\cos\phi} = \frac{298}{\sqrt{f}} \log_{10} \frac{1 + \cos(\theta - \phi)}{2\cos\theta\cos\phi} \tag{20}$$

in which A_C is, as before, the attenuation constant of standard cable. If $f = 796$ cycles (or $\omega = 5{,}000$), equation (20) becomes

$$L_O = 10.56 \log_{10} \frac{1 + \cos(\theta - \phi)}{2\cos\theta\cos\phi} \tag{21}$$

Equation (21) is plotted in Fig. 3.

As an illustration of the use of the above equations in determining the transformer and the transition losses existing at any point in a circuit, let us consider the circuit shown in Fig. 4. The ratio (r) of the impedances at the point under consideration is evidently

$$r = \frac{1{,}500 \underline{/55^\circ}}{500 \underline{/20^\circ}} = 3 \underline{/75^\circ}$$

Consequently, $r = 3$ and $\beta = 75°$. From the curves in Fig. 2 it is seen that the insertion of an ideal transformer at this point would cause a gain in efficiency of the circuit amounting to 1.94 miles. Moreover, from the curves in Fig. 3—in which θ is taken as $\overline{/20°}$ (or $-20°$) and ϕ as $\underline{/55°}$ (or $+55°$)—it is found that we must add .70 mile to the transformer loss in order to get the transition loss at the junction under consideration. Hence, the transition loss in the case assumed is $1.94 + .70 = 2.64$ miles of standard cable. This latter value is, therefore, the maximum amount by which the efficiency of the circuit can be improved, regardless of the nature of the passive network which might be inserted at the junction under consideration.

From the definition of A_O and from (14) and (19) we get

$$A_{\text{TRT.}} = A_{\text{TRF.}} + A_O = 1.1513 \log_{10} \frac{1 + 2r \cos\beta + r^2}{2r(1 + \cos\beta)} + 1.1513 \log_{10} \frac{1 + \cos\beta}{2 \cos\theta \cos\phi} \tag{22}$$

or

$$A_{\text{TRT.}} = 1.1513 \log_{10} \frac{1 + 2r \cos\beta + r^2}{4r \cos\theta \cos\phi} \tag{23}$$

Similarly, expressing the transition loss $L_{\text{TRT.}}$ in miles of standard cable, we have

$$L_{\text{TRT.}} = L_{\text{TRF.}} + L_O = \frac{298}{\sqrt{f}} \log_{10} \frac{1 + 2r \cos\beta + r^2}{2r(1 + \cos\beta)} + \frac{298}{\sqrt{f}} \log_{10} \frac{1 + \cos\beta}{2 \cos\theta \cos\phi} \tag{24}$$

or

$$L_{\text{TRT.}} = \frac{298}{\sqrt{f}} \log_{10} \frac{1 + 2r \cos\beta + r^2}{4r \cos\theta \cos\phi} \tag{25}$$

Also at 796 cycles ($\omega = 5{,}000$) we get from (24) and (25)

$$L_{\text{TRT.}} = 10.56 \log_{10} \frac{1 + 2r \cos\beta + r^2}{2r(1 + \cos\beta)} + 10.56 \log_{10} \frac{1 + \cos\beta}{2 \cos\theta \cos\phi} \tag{26}$$

or

$$L_{\text{TRT.}} = 10.56 \log_{10} \frac{1 + 2r \cos\beta + r^2}{4r \cos\theta \cos\phi} = 10.56 \log_{10} \frac{1 + 2r \cos(\theta - \phi) + r^2}{4r \cos\theta \cos\phi} \tag{27}$$

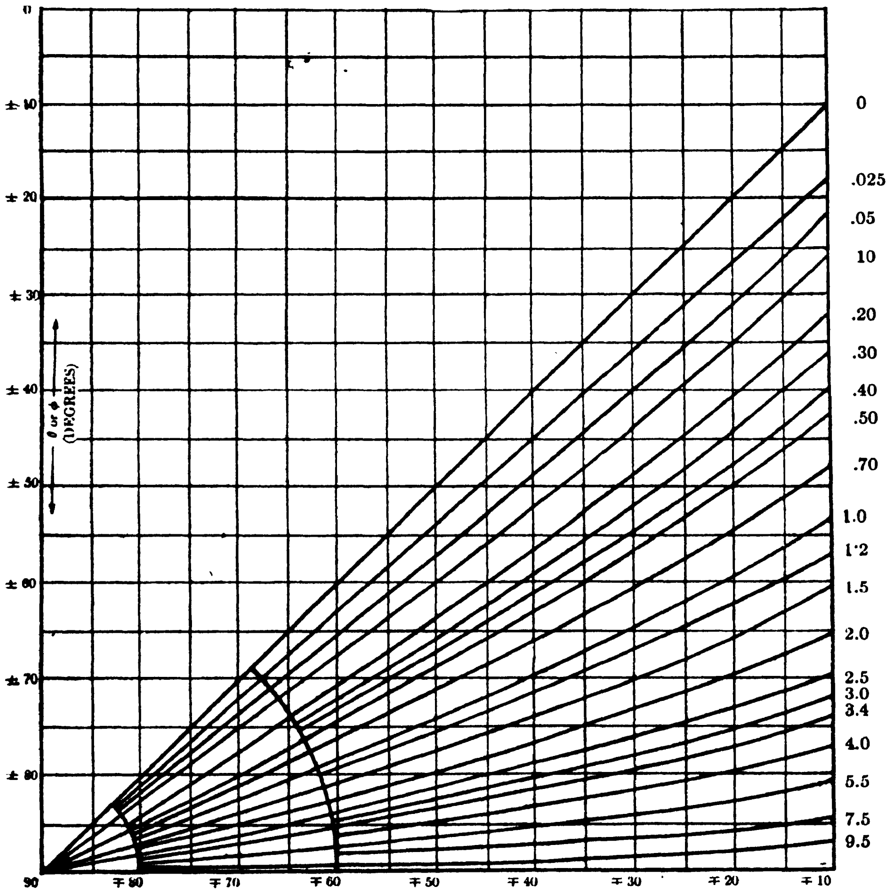

FIG. 3. Phase difference loss curves—or curves showing the magnitude of the gain which when added to the gain obtained by the insertion of an ideal transformer of the optimum ratio gives the value of the maximum gain it is possible to obtain at any point in a telephone circuit by the insertion of a passive network. Computed for a frequency of 796 cycles.

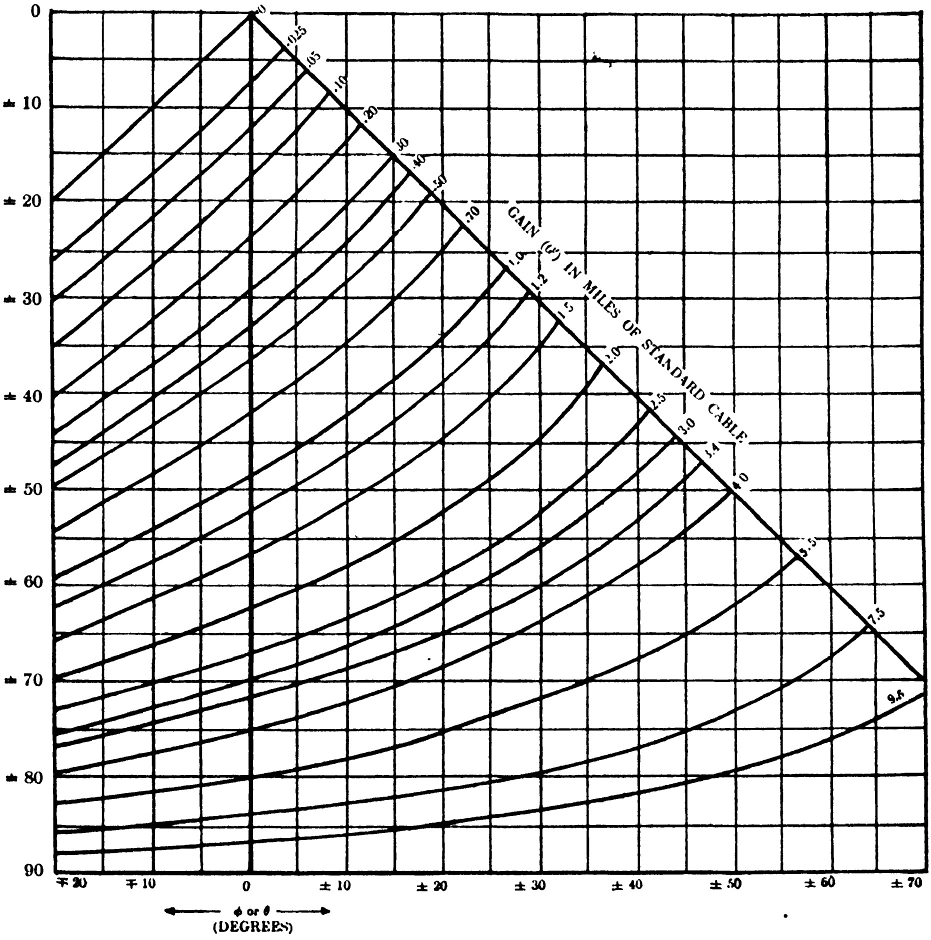

FIG. 3 (*Continued*).

7.2 Determination of the Best Terminal Impedances for Any Given Structure.—It is frequently desirable to be able to determine what is the *minimum* transmission loss which *must* be encountered in a telephone circuit—by employing a given network in that circuit—as compared with the loss in a properly designed circuit which does *not* include the given network.

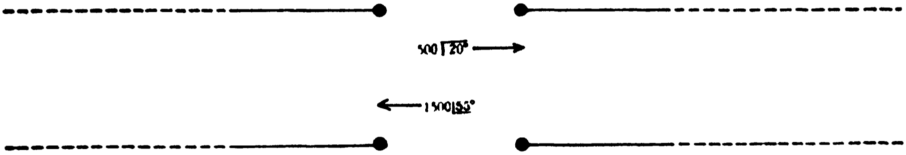

FIG. 4. Circuit illustrating the use of the terms *transformer loss* and *transition loss.*

Suppose we have a circuit (Fig. 5) in which Z_A, Z_B and Z_C are the impedances of the three arms of a T network. It can be proved * that such a T network can, at any given frequency, represent any passive bilateral device such as a transformer, a length of line, etc. Formulae for reducing various pieces of apparatus to such an equivalent T network are given in Sect. V of Appendix D.

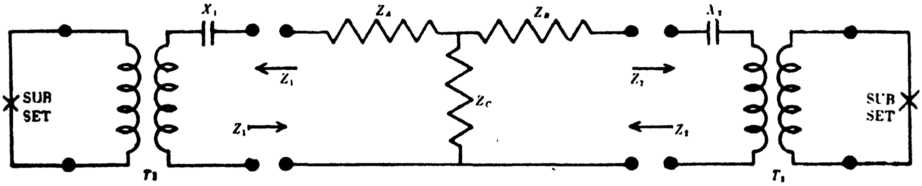

FIG. 5. Dissymmetrical T network connected between reactances and transformers of such constants as to cause no transition losses at the junctions of the T network and the rest of the circuit.

If we connect the substation sets by ideal transformers T_1 and T_2 and by series reactances X_1 and X_2 and so adjust the ratios of these transformers and the values of the reactances that the impedances, $\overline{Z}_1$ and $\overline{Z}_2$, † looking into the T network are the conjugates respectively of the impedances Z_1 and Z_2 looking into the sets, we have then eliminated all the transition losses existing in the circuit—on the assumption that we can not change the efficiency or constants of the substation sets or the construction of the T network under consideration. Under the above conditions, the circuit is therefore as efficient as it can be if we are required to use the given substation sets and the given T network.

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* See G. A. Campbell's paper on "Cisoidal Oscillations," *Transactions of the A. I. E. E.*, Vol. XXX, 1911, Part II, pp. 873–909. Also see Chapter VIII.

† A dash over a symbol is employed throughout to indicate that the quantity represented is the conjugate of the quantity which would be represented by the symbol without the dash.

Suppose now that the T network were eliminated from the circuit and that we were again to design the circuit so that it would still have the maximum possible efficiency. We would then adjust the ratios of the transformers and the values of the reactances so that the impedances Z_1 and Z_2 would be the conjugates of each other, thereby eliminating the transition loss at the junction of the two substation sets.

A comparison of the received currents in the two cases—with the T network in and with the T network out—would then give us a measure of the minimum transmission loss which *must* be encountered in a telephone circuit by employing the given T network.

7.21 Dissymmetrical (T) Network—General Case.—Let us consider then the case of a dissymmetrical T network which can, as will be demonstrated in Chapter VIII, represent any composite circuit or any piece of apparatus (except repeater elements and their like) and obtain expressions for the best values of the terminal impedances—these expressions being in terms of the effective resistances and reactances of the various arms of the T network and being such that there will be no transition losses existing at the junctions of the network and the terminal impedances.*

Consider then the circuit shown in Fig. 6. If there are to be no transition losses at the terminals of the T network, we evidently must have the relations

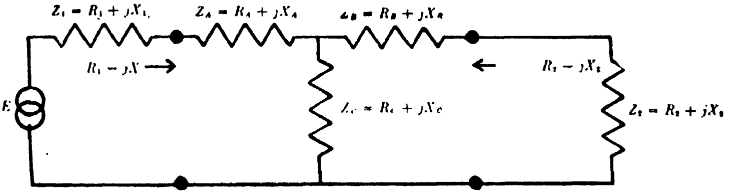

FIG. 6. Dissymmetrical T network connected between generalized terminal impedances of such values as to cause no transition losses.

$$\overline{Z}_1 = Z_A + \frac{Z_C(Z_B + Z_2)}{Z_C + Z_B + Z_2} \tag{28}$$

and

$$\overline{Z}_2 = Z_B + \frac{Z_C(Z_A + Z_1)}{Z_C + Z_A + Z_1} \tag{29}$$

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* The solution of the problem discussed in this section was first given by P. B. Flanders.

Taking the conjugates of each side of the last equation, we have

$$Z_2 = \bar{Z}_B + \frac{\bar{Z}_C(\bar{Z}_A + \bar{Z}_1)}{\bar{Z}_C + \bar{Z}_A + \bar{Z}_1} \tag{30}$$

Putting in this last expression the value of $\bar{Z}_1$ as given by (28), we have

$$Z_2 = \bar{Z}_B + \frac{\bar{Z}_C\left[\bar{Z}_A + Z_A + \dfrac{Z_C(Z_B + Z_2)}{Z_C + Z_B + Z_2}\right]}{\bar{Z}_C + \bar{Z}_A + Z_A + \dfrac{Z_C(Z_B + Z_2)}{Z_C + Z_B + Z_2}} \tag{31}$$

or

$$Z_2 = \bar{Z}_B + \frac{Z_C[(\bar{Z}_A + Z_A + Z_C)Z_2 + (Z_B + Z_C)(\bar{Z}_A + Z_A) + Z_BZ_C]}{Z_2(Z_A + \bar{Z}_A + Z_C + \bar{Z}_C) + (Z_A + \bar{Z}_A + \bar{Z}_C)(Z_B + Z_C) + Z_BZ_C} \tag{32}$$

Whence

$$\begin{aligned} Z_2^2[Z_A + \bar{Z}_A + Z_C + \bar{Z}_C] & \\ + Z_2[(Z_A + \bar{Z}_A)(Z_B - \bar{Z}_B + Z_C - \bar{Z}_C) + (Z_C + \bar{Z}_C)(Z_B - \bar{Z}_B)] & \\ - [(Z_A + \bar{Z}_A)(Z_B + Z_C)(\bar{Z}_B + \bar{Z}_C) & \\ + Z_B\bar{Z}_B(Z_C + \bar{Z}_C) + Z_C\bar{Z}_C(Z_B + \bar{Z}_B)] = 0 & \end{aligned} \tag{33}$$

Solving this quadratic equation and expressing the results in terms of the effective resistance and reactance components of the various impedances involved, we get:

$$Z_2 = \frac{R_AR_B + R_BR_C + R_AR_C}{R_A + R_C}\sqrt{1 + \frac{R_C^2 + X_C^2}{R_AR_B + R_BR_C + R_AR_C}} - j\left[X_B + \frac{R_AX_C}{R_A + R_C}\right] \tag{34}$$

Similarly from the symmetry of the structure under consideration, we evidently have:

$$Z_1 = \frac{R_AR_B + R_BR_C + R_AR_C}{R_B + R_C}\sqrt{1 + \frac{R_C^2 + X_C^2}{R_AR_B + R_BR_C + R_AR_C}} - j\left[X_A + \frac{R_BX_C}{R_B + R_C}\right] \tag{35}$$

As a numerical example showing the use of the above formulae, let us consider a typical local battery induction coil and determine, first, between what impedances it will work most efficiently and, secondly, what is the minimum loss caused by such an induction coil—on the assumption of ideally designed circuits. We will assume the following 800 cycle constants for the self and mutual impedances of the coil:

$$Z_P = 3.1 + j24$$
$$Z_S = 36.6 + j413$$
$$Z_M = 3.7 + j93.5$$

From the well-known formulae (see Fig. 23B, Appendix D) for reducing a transformer to its equivalent T network, the induction coil under consideration can be reduced to the equivalent T network shown in Figure 7

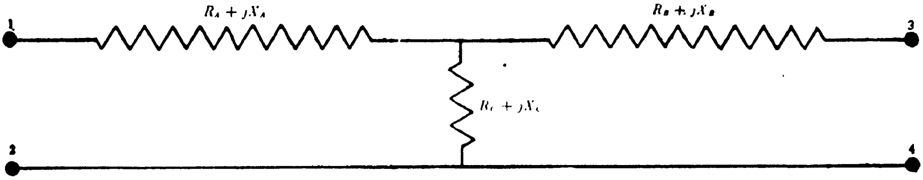


FIG. 7. Equivalent T network, representing a local battery induction coil at 800 cycles.

in which

$$R_A = -.6 \qquad R_B = 32.9 \qquad R_C = 3.7$$
$$X_A = -69.5 \qquad X_B = 319.5 \qquad X_C = 93.5$$

Putting these values in equations (34) and (35), we get

$$Z_1 = 25.7 - j14.5$$

and

$$Z_2 = 303 - j301.4$$

If, then, we consider the induction coil to be connected to terminal impedances having the values given above, we have the circuit shown in Fig. 8, in which the absolute magnitude of the computed current (see formula (3)) in one branch with an electromotive force E acting in the other branch is

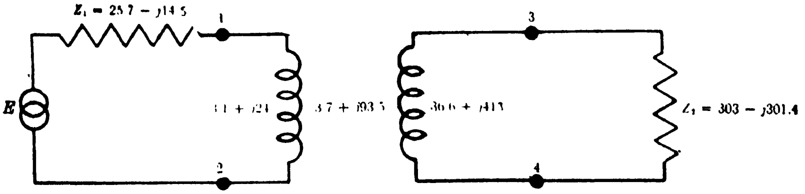


FIG. 8. Induction coil connected between terminal impedances of such values as to result in no transition loss—either at the points 1–2 or 3–4.

$$|I| = .00509E$$

If there were no transformer in the circuit but if the two impedances were connected directly to each other, the current, $E/(Z_1 + Z_2)$, would have been

$$|I| = .00219E$$

If the two impedances had been connected together by an ideal transformer of the best possible impedance ratio, the current (computed by formula (8)) would have been

$$|I| = .00449E$$

while, if the impedances had been connected together by an ideal transducer, the current (computed by formula (9)) would have been

$$|I| = .00567E$$

Assuming a frequency of 796 cycles ($\omega = 5{,}000$), it may be shown from these currents that, in the above circuit, (1) the introduction of the induction coil has caused a gain of 7.75 miles of standard cable, (2) the efficiency of the induction coil is actually 1.15 miles of standard cable *above* that of the ideal transformer in efficiency and (3) the efficiency of the induction coil is only 1.0 mile *below* that of an ideal transducer. In other words, *it is possible for a telephone transformer that is ordinarily regarded as relatively inefficient, such as a local battery induction coil, to be actually more efficient, when connecting together certain impedances, than would be an ideal transformer at the same junction.* This is never possible, however, unless one or both of the reactances of the branch circuits are negative—in which case these reactances may be effectively annulled by the positive or inductive reactances of the transformer.

7.22 Symmetrical (*T*) Network.—Let us now consider the case of a symmetrical *T* network, such as is shown in Fig. 9.

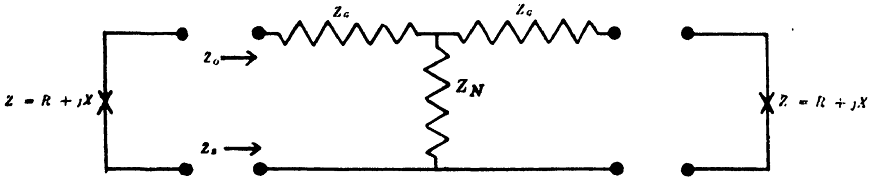


FIG. 9. Symmetrical *T* network connected between terminal impedances of such value as to cause no transition losses.

Let Z_O be the open-circuited impedance of the *T* network, or

$$Z_O \equiv Z_G + Z_N \tag{36}$$

Similarly let Z_S be the impedance of the *T* network with its distant end short-circuited, or

$$Z_S \equiv Z_G + \frac{Z_G Z_N}{Z_G + Z_N} \tag{37}$$

Solving these two equations simultaneously,

$$Z_N = \sqrt{Z_O^2 - Z_O Z_S} \tag{38}$$

and

$$Z_G = Z_O - \sqrt{Z_O^2 - Z_O Z_S} \tag{39}$$

If we let $Z \equiv R + jX$ be the value of one terminal impedance to which the T network is to be connected, it is evident from the symmetry of the network itself that the other terminal impedance will also be Z.

Let $\overline{Z}$ be the conjugate of Z, or $\overline{Z} \equiv R - jX$. If there is to be no transition loss at the junction of the network and the terminal impedance Z, it is evident that we must have

$$\overline{Z} = Z_G + \frac{Z_N(Z_G + Z)}{Z_G + Z_N + Z} \tag{40}$$

or

$$\overline{Z}(Z_G + Z_N) + Z\overline{Z} = Z_G(Z_G + Z_N) + Z_G Z + Z_N(Z_G + Z) \tag{41}$$

Putting the values of Z_N and Z_G as given by (38) and (39) in (41), we get

$$\overline{Z}Z + \overline{Z}Z_O - ZZ_O - Z_O Z_S = 0 \tag{42}$$

If we let $Z_O \equiv R_O + jX_O$ and $Z_S \equiv R_S + jX_S$, (42) gives

$$\begin{aligned}(R - jX)(R + jX) &+ (R - jX)(R_O + jX_O) \\ &- (R + jX)(R_O + jX_O) - (R_O + jX_O)(R_S + jX_S) = 0\end{aligned} \tag{43}$$

or

$$\begin{aligned}(R^2 + X^2 &+ 2XX_O - R_O R_S + X_O X_S) \\ &- j(R_O X_S + R_S X_O + 2R_O X) = 0\end{aligned} \tag{44}$$

Equating the imaginary part of (44) to zero, we get

$$X = -\frac{R_O X_S + R_S X_O}{2R_O} = -\frac{1}{2}\left(X_S + \frac{R_S X_O}{R_O}\right) \tag{45}$$

Similarly, equating the real part of (44) to zero,

$$\begin{aligned}R^2 + X^2 &= R_O R_S - X_O X_S - 2XX_O \\ &= R_O R_S - X_O X_S - 2X_O\left[-\frac{1}{2}\left(X_S + \frac{R_S X_O}{R_O}\right)\right]\end{aligned} \tag{46}$$

or

$$R^2 + X^2 = R_O R_S + \frac{R_S X_O^2}{R_O} = \frac{R_S}{R_O}(R_O^2 + X_O^2) \tag{47}$$

Whence

$$R = \sqrt{\frac{R_S}{R_O}(R_O^2 + X_O^2) - X^2}$$
$$= \sqrt{\frac{R_S}{R_O}(R_O^2 + X_O^2) - \frac{1}{4}\left[X_S + \frac{R_S X_O}{R_O}\right]^2} \tag{48}$$

If then we are given a symmetrical T network with known values for the impedances of the arms, Z_G and Z_N, and wish to determine the value Z of the terminal impedances which will result in the maximum transfer of power through the T network, we can first determine Z_O and Z_S as given by (36) *and* (37) *and then determine X and R by means of* (45) *and* (48). *Whence Z is determined—since $Z \equiv R + jX$.*

In order to illustrate the use of such formulae, let us determine what is the minimum transmission loss that will be encountered in an ideally designed circuit due to the insertion of a 1 mile length of standard cable. The equivalent T network of such a length of cable at 796 cycles is as shown in Fig. 10, or $Z_G = 44 + j0$ and $Z_N = 0 - j3{,}700$. From formulae (36) and (37) we have $Z_O = 44 - j3{,}700$ and $Z_S = 88 - j.5$ or $R_O = 44$, $X_O = -3{,}700$, $R_S = 88$ and $X_S = -.5$. Putting these values in (45) and (48), we get $X = 3{,}700$ and $R = 3{,}700$. A maximum transfer of power through such a length of cable will, therefore, be obtained when it is working between two equal impedances whose phase angles are approximately + 45°. The effective resistance and effective reactance of each of these impedances, moreover, will be equal to the absolute magnitude of the shunt impedance of the equivalent T network of the cable. Assuming that the given T network is connected between the best possible terminal impedances of $3{,}700 + j3{,}700$ (see Fig. 11) and that an e.m.f. E is acting in one of the impedances, the current in the other impedance is $|I| = E/7{,}488$. If the cable had not been inserted and the reactance of the terminal impedances had been annulled, the current flowing in the distant impedance would have been $|I| = E/7{,}400$.

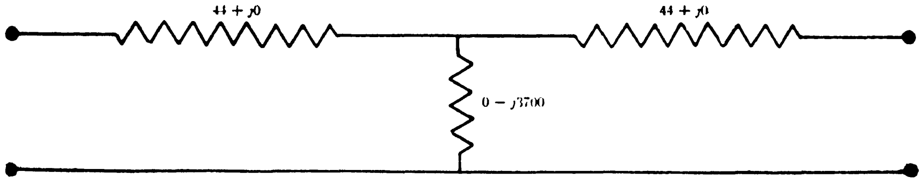


FIG. 10. Equivalent T network representing one mile of No. 19 gauge (88 ohms, .054 mf.) cable at 796 cycles.

The ratio of these two currents is 7,400/7,488 = .988. This corresponds to .12 mile of cable at 796 cycles. In other words, the insertion of a mile of standard cable in the case of two ideally designed circuits will, at 796 cycles, cause a transmission loss of only .12 mile.

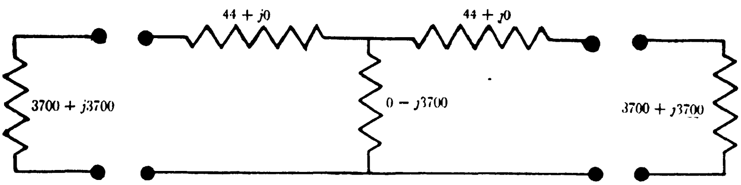


FIG. 11. Equivalent *T* network, representing one mile of standard cable, connected between the best possible terminal impedances.

The insertion of a mile of standard cable may, of course, in any given case actually cause an appreciable gain instead of a loss. Suppose, for example, that in the case under consideration we had compared the current flowing ($I = E/7{,}488$) with the *T* network inserted in the circuit as compared with the current flowing when the *T* network was omitted. The current in the latter case would have been $E/(7{,}400 + j7{,}400)$ or $E/10{,}460$. The ratio of this current to that flowing when the mile of cable was in the circuit is 7,488/10,460 = .715 which corresponds to 3.1 miles of standard cable at 796 cycles. In other words, at 800 cycles, the insertion of a mile of standard cable would actually have caused a *gain* of approximately 3 miles of standard cable. The fact that the insertion of a given length of cable can actually cause a gain, of greater value than the length of the cable inserted, is well known. A particularly familiar example of this sort is exhibited by the characteristics of the so-called *system reference standard.* In this system, the insertion of two or three miles of standard cable in the trunk circuit does not actually produce a loss in the efficiency of the circuit but on the other hand produces a slight gain so that a system whose *transmission equivalent* is three miles is actually more efficient than a system whose equivalent is zero. A curve showing the relation between the transmission equivalent and the transmission loss in the system reference standard is shown in Fig. 12. It is to be noted that this curve, which is the result of computations made at 800 and 1,500 cycles, does not become linear until approximately 8 miles of standard cable have been added to the circuit.

7.23 Dissymmetrical *T* Network Whose Arms are Pure Resistances.—Let us next consider the special case of a dissymmetrical *T* network all of whose arms are pure resistances (see Fig. 13). If the terminal

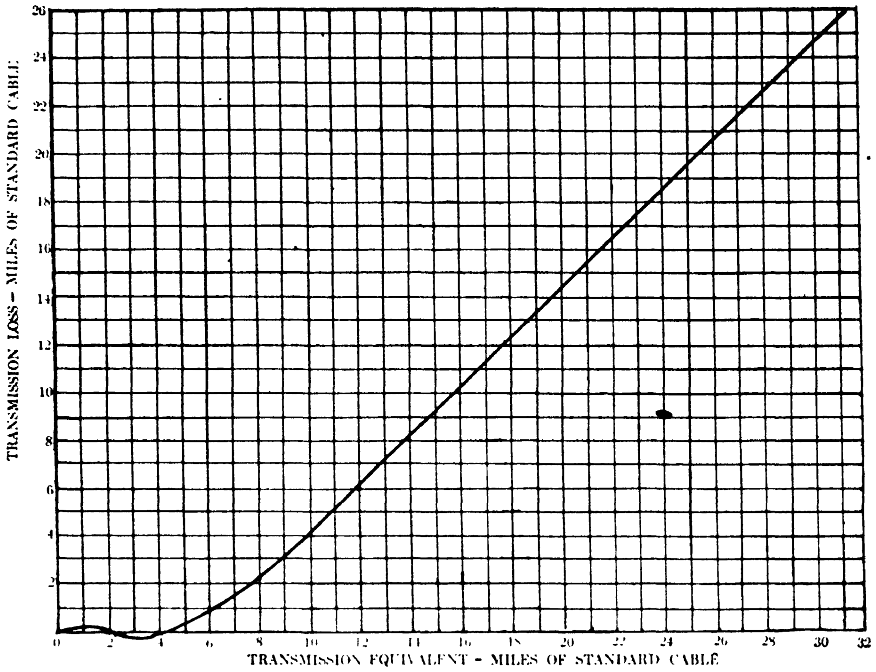


FIG. 12. Relation between transmission equivalent and transmission loss in the system reference standard.

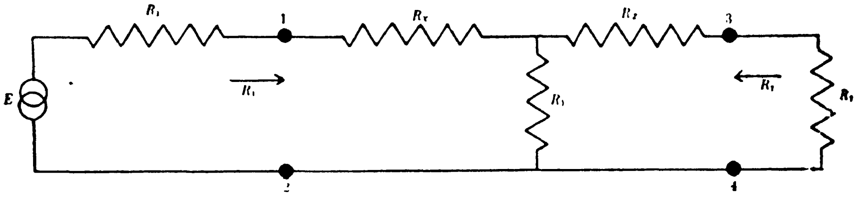


FIG. 13. Dissymmetrical *T* network whose arms are composed of pure resistances and connected between terminal resistances of such values as to eliminate all transition losses at the junction points 1–2 and 3–4.

impedances R_1 and R_2—which in this case will be pure resistances—are to be such that there will be no transition losses at the terminals 1–2 and 3–4, we evidently will have

$$R_1 = R_X + \frac{R_Y(R_2 + R_Z)}{R_Y + R_2 + R_Z} \tag{49}$$

and

$${}_2 = R_Z + \frac{R_Y(R_1 + R_X)}{R_Y + R_1 + R_X} \tag{50}$$

Solving these equations simultaneously, we have for the optimum values of the terminal impedances

$$R_1 = \sqrt{\frac{(R_X + R_Y)(R_XR_Y + R_YR_Z + R_XR_Z)}{R_Y + R_Z}} \tag{51}$$

and

$$R_2 = \sqrt{\frac{(R_Y + R_Z)(R_XR_Y + R_YR_Z + R_XR_Z)}{R_X + R_Y}} \tag{52}$$

Since the open- and short-circuit impedances of the given T network from the 1–2 end are respectively

$$Z_O = R_X + R_Y \tag{53}$$

and

$$Z_S = \frac{R_XR_Y + R_YR_Z + R_XR_Z}{R_Y + R_Z} \tag{54}$$

and the corresponding impedances from the 3–4 end are

$$Z_O' = R_Y + R_Z \tag{55}$$

and

$$Z_S' = \frac{R_XR_Y + R_YR_Z + R_XR_Z}{R_X + R_Y} \tag{56}$$

we can rewrite equations (51) and (52) as follows:

$$R_1 = \sqrt{Z_OZ_S} \tag{57}$$

and

$$R_2 = \sqrt{Z_O'Z_S'} \tag{58}$$

These latter equations give a ready means for determining the optimum values of the terminating impedances when the open- and short-circuit impedances of any structure, which is comprised simply of pure resistances, are known.

If we take a special case of the above in which the R_Z arm is of zero resistance, the circuit reduces to that shown in Fig. 14. In this circuit

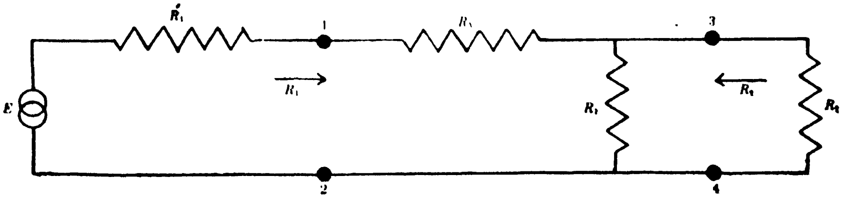


FIG. 14. *L* network whose arms are composed of pure resistances of such values as to eliminate all transition losses at the junction points 1–2 and 3–4.

let us consider the problem of determining the minimum difference in transmission efficiency which *must* exist between two telephone circuits each properly designed for maximum efficiency—one of which includes and the other of which does not include the L-shaped network (1–2–3–4) shown in Fig. 14.

It is seen from formulae (51) and (52) that a network having resistance arms R_X and R_Y, as shown above, will cause a minimum transmission loss in a circuit of the above type when the transformer ratios are such that

$$R_1 = \sqrt{R_X(R_X + R_Y)} \tag{59}$$

and

$$R_2 = R_Y\sqrt{\frac{R_X}{R_X + R_Y}} \tag{60}$$

If we change the notation slightly and let Kr (instead of R_X) be the series arm of the network and let r be the resistance of the shunt arm (instead of R_Y), we have the circuit shown in Fig. 15.

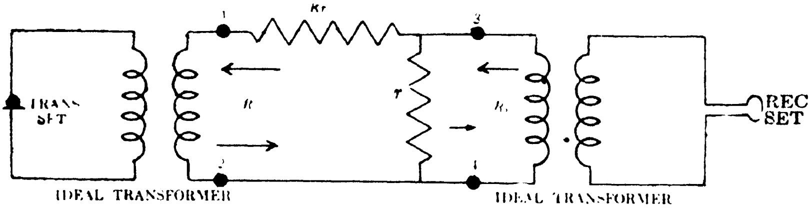


FIG. 15. L network whose arms are composed of pure resistances of such values as to eliminate all transition losses at the junction points 1–2 and 3–4 (slightly different notation from Fig. 14).

Formulae (59) and (60) now become

$$R_1 = \sqrt{Kr(Kr + r)} = r\sqrt{K}\,\sqrt{K + 1} \tag{61}$$

$$R_2 = \frac{r\sqrt{Kr}}{\sqrt{Kr + r}} = \frac{r\sqrt{K}}{\sqrt{K + 1}} \tag{62}$$

The ratio of the currents in the receiver after and before inserting the network (1–2–3–4) is

$$\frac{I_2}{I_1} = \frac{r(R_1 + R_2)}{(R_1 + Kr)(R_2 + r) + R_2 r} \tag{63}$$

Putting in (63) the values of R_1 and R_2 as given in (61) and (62), we get

$$\frac{I_2}{I_1} = \frac{K + 2}{1 + (\sqrt{K} + \sqrt{K + 1})^2} \tag{64}$$

If we assume a frequency of 796 cycles, the minimum transmission loss—in miles of standard cable—which the insertion of the network will cause in the above circuit is

$$L_1 = 21.13 \log_{10} \frac{1 + (\sqrt{K} + \sqrt{K+1})^2}{K+2} \tag{65}$$

It will be noted that if K is very large, the above expression reduces to

$$L_1 = 21.13 \log_{10} 4 = 12.7 \text{ miles} \tag{66}$$

On the other hand, if the network were not to be inserted in the circuit and if we were to have maximum transmission efficiency, the impedances R_1 and R_2 should be equal. Hence, in order to determine what is the minimum transmission loss which *must* be encountered by employing such a network in a circuit—as compared with the loss in a properly designed circuit which does not include the above network—we must add to the value of L_1, as given by (65), the transformer loss corresponding to the ratio of R_1/R_2. From (61) and (62) it is seen that this ratio is

$$\frac{R_1}{R_2} = K + 1 \tag{67}$$

The transformer loss corresponding to this ratio may be obtained from curves such as are shown in Fig. 2, or assuming a frequency of 796 cycles, it may be obtained from the formula

$$L_2 = 21.13 \log_{10} \frac{K+2}{2\sqrt{K+1}} \tag{68}$$

Consequently, the sum of (65) and (68), which is

$$\begin{aligned} L = L_1 + L_2 = 21.13 \log_{10} &\frac{1 + (\sqrt{K} + \sqrt{K+1})^2}{2\sqrt{K+1}} \\ &= 21.13 \log_{10} \left[\sqrt{K+1} + \sqrt{K}\right] \end{aligned} \tag{69}$$

gives us the minimum number of miles of standard cable (at 796 cycles) by which the efficiency of any properly designed telephone circuit that is required to include the network, as shown, will be below that of a similar circuit also designed for maximum efficiency, but not including the network.

For the sake of reference, there is given in Table I a set of values plotted from (69):

TABLE I

K	L	K	L
0	0	100	27.5
1	8.1	200	30.7
10	17.1	300	32.5
25	21.2	500	34.9
50	24.4	1000	38.0

We have seen that for large values of K the value of L_1 (at 796 cycles) approaches 12.7 miles of standard cable. Hence, for large values of K, the value of L—corresponding to any given value of K—is approximately 12.7 miles plus the *transformer loss*, corresponding to the resistance ratio $(K + 1)$.

As a typical numerical example of the above, suppose that we had a network which had a series resistance of 100 ohms and a shunt resistance of 1 ohm (see Fig. 16).

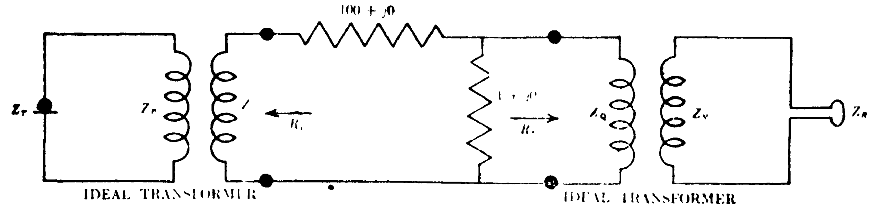


FIG. 16. L network connected between its optimum terminal impedances.

Using the notation which we have employed, K is equal to 100 and $r = 1$. Whence we find from formulae (61) and (62) that in order for the network to cause a minimum transmission loss, the transformer ratios must be such that

$$R_1 = \frac{Z_S}{Z_P} Z_T = \sqrt{100 \times 101} = 100.5 \text{ ohms} \tag{70}$$

and

$$R_2 = \frac{Z_Q}{Z_N} Z_R = \sqrt{\frac{100}{101}} = .994 \text{ ohm} \tag{71}$$

With transformers of these ratios, we find from (65) that the insertion of the network in the above circuit causes a transmission loss of

$$L_1 = 21.13 \log_{10} \frac{1 + (\sqrt{100} + \sqrt{101})^2}{100 + 2} = 12.6 \text{ miles} \tag{72}$$

which is within .1 of a mile of the maximum value which L_1 can ever have. If the circuit did not include the network, however, the resistances R_1 and R_2 would be equal to each other. Hence, we must add the transformer loss corresponding to the ratio of $R_1/R_2 = K + 1 = 101$. This loss, as given by (68), is

$$L_2 = 21.13 \log_{10} \frac{102}{2\sqrt{101}} = 14.9 \text{ miles} \tag{73}$$

The sum of the values of L_1 and L_2 as given by (72) and (73) is 27.5 miles which is identical with that given by equation (69) and listed in Table I.

It is thus shown that any circuit which includes the particular network chosen—a network whose series resistance is 100 times its shunt resistance—must have an efficiency which is *at least* 27.5 miles (at 796 cycles) below that of a similar circuit, properly designed, but which does not include the network. This loss is very closely equal to 12.7 miles plus the transformer loss corresponding to the ratio of the resistance of the series arm to that of the shunt arm of the network.

Equation (69) *and the tabular values which have been computed from it give a very ready means of determining approximately the minimum transmission loss which any given L network of pure resistances will introduce into a properly designed circuit.*

If one is interested in any frequency f, other than 796 cycles, it can easily be shown that the general formulae corresponding to (65), (68) and (69) are respectively

$$L_1 = \frac{596}{\sqrt{f}} \log_{10} \frac{1 + (\sqrt{K} + \sqrt{K+1})^2}{K+2} \tag{74}$$

$$L_2 = \frac{596}{\sqrt{f}} \log_{10} \frac{K+2}{2\sqrt{K}+1} \tag{75}$$

and

$$L = L_1 + L_2 = \frac{596}{\sqrt{f}} \log_{10} (\sqrt{K} + 1 + \sqrt{K}) \tag{76}$$

7.3 The Equivalent T Network of an Ideal Transducer or Transformer.—We have already shown in Sect. 7.1 that if we have given a circuit whose sending end impedance is $R_1 + jX_1$ and whose receiving end impedance is $R_2 + jX_2$, we will be connecting these impedances by an *ideal transducer* provided we first annul the reactances jX_1 and jX_2 of the terminal impedances by the insertion of equal negative reactances $-jX_1$ and $-jX_2$ in series with them (see Fig. 17) and then insert (between the terminals 1–2 and 3–4) a T network that is *equivalent to* that of an ideal transformer of the optimum ratio. In other words, under such

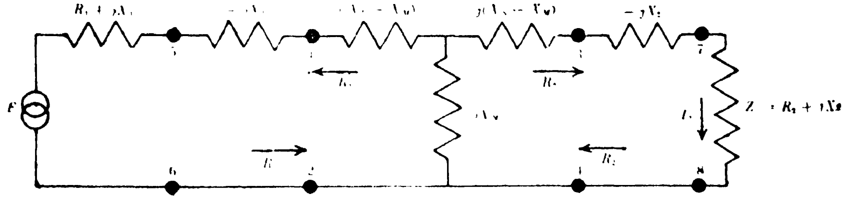


FIG. 17. T network (having terminals 1–2 and 3–4) composed of pure reactances and equal in transmission efficiency to an ideal transformer of the optimum impedance ratio when connected between terminal impedances (R_1 and R_2) having no reactances.

conditions the maximum possible energy ($E^2/4R_1$) will be absorbed in the receiving end impedance, Z_2.

In Fig. 17 we have between terminals 1, 2, 3 and 4 a T network whose arms are assumed to be made up of pure reactances. If there is to be no transition loss either at the junction 1–2 or at the junction 3–4, we evidently must have the relations

$$R_1 = j(X_P - X_M) + \frac{jX_M[R_2 + j(X_S - X_M)]}{R_2 + jX_S} \tag{77}$$

and

$$R_2 = j(X_S - X_M) + \frac{jX_M[R_1 + j(X_P - X_M)]}{R_1 + jX_P} \tag{78}$$

From (77) we can derive the relations*

$$\frac{R_1}{R_2} = \frac{X_P}{X_S} \tag{79}$$

and

$$X_M^2 = X_P X_S + R_1 R_2 \tag{80}$$

These latter relations give us the equivalent T network shown in Fig. 18,

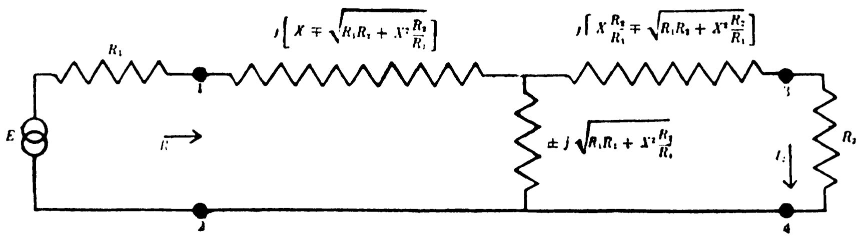

R_1
$j\left[X \mp \sqrt{R_1R_2 + X^2\frac{R_2}{R_1}}\right]$
$j\left[X\frac{R_2}{R_1} \mp \sqrt{R_1R_2 + X^2\frac{R_2}{R_1}}\right]$
1
3
E
I_1
$\pm j\sqrt{R_1R_2 + X^2\frac{R_2}{R_1}}$
I_2
R_2
2
4

FIG. 18. Ideal transducer connected between terminal impedances whose reactances have been annulled.

which evidently represents an ideal transformer, or an ideal transducer when working between pure resistance terminal impedances—insofar as the value of $|I_2|$ is concerned.

Between the terminals 1–2 and 3–4 in the network shown in Fig. 18, jX is a variable pure reactance that may arbitrarily be given any desired value between zero and infinity without affecting the value of $|I_2|$. If X is infinitely large, the network of Fig. 18 reduces to that shown in Fig. 19—the T network of the ordinary well-known type of transformer.

Under this condition the make-up of the arms of the T network is simply dependent upon the *ratio* R_2/R_1 and the phase of I_2 is either 0° or 180° out of phase with E.

---

* These equations were first derived by R. L. Wegel.

If, in Fig. 18, we assume the variable X to be zero, we find that the arms of the T network are a function simply of the *product* of R_1 and R_2 (instead of being a function of their ratio as in the preceding case), the network reducing to the structure shown in Fig. 20. In this case, the phase of I_2 is ± 90° out of phase with E.

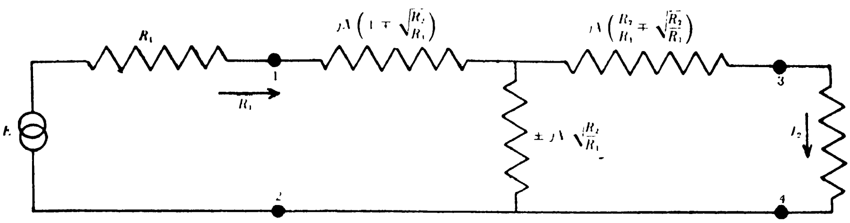

R_1
$jX\left(1 \mp \sqrt{\frac{R_2}{R_1}}\right)$
$jX\left(\frac{R_2}{R_1} \mp \sqrt{\frac{R_2}{R_1}}\right)$
1
I_1
3
E
$\pm jX\sqrt{\frac{R_2}{R_1}}$
I_2
R_2
2
4

FIG. 19. Equivalent T network of an ideal two-winding transformer of the ordinary type connected between optimum terminal resistances.

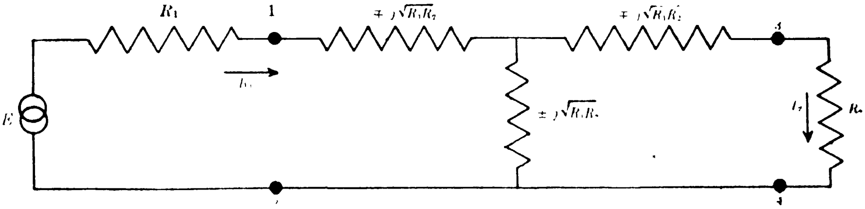


FIG. 20. T network (comprised of reactances, *equal numerically* to the square root of the product of the terminal resistances) which is equivalent to an ideal transformer.

Finally, if we let $X^2 = R_1R_2 + X^2(R_2/R_1)$, the left arm of the general type of T network (in Fig. 18) becomes zero and the structure then reduces to the two-arm or L type of network shown in Fig. 21. In this latter case, the make-up of the ideal transformer is seen to depend both upon the ratio and upon the product of R_1 and R_2. Similarly, the phase of I_2 is, in general, between 0° and ± 90° or between ± 90° and ± 180° out of phase with E—depending upon the relative magnitudes of R_1 and R_2.

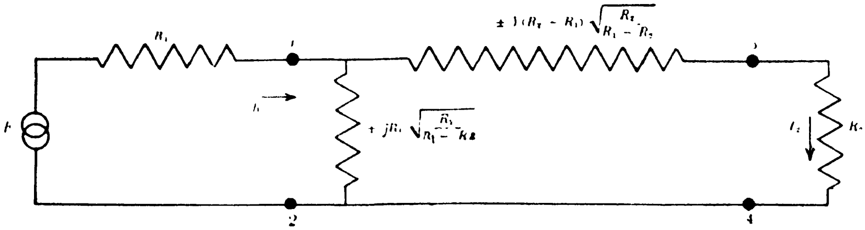

$\pm j(R_2 - R_1)\sqrt{\frac{R_2}{R_1 - R_2}}$
R_1
1
3
I_1
E
$\mp jR_1\sqrt{\frac{R_2}{R_1 - R_2}}$
I_2
R_2
2
4

FIG. 21. L network comprised of pure reactances of finite values and equivalent to an ideal transformer.

From Fig. 21 it is apparent that we can always make up an ideal transducer out of not more than four reactances, two of which, $-jX_1$ and $-jX_2$, may be used to annul the reactances $+jX_1$ and $+jX_2$ of the terminal impedances and the other two—as shown in the L network of the above figure—to function as the equivalent of an ideal transformer of the proper impedance ratio, insofar as the magnitude of I_2 is concerned. Suppose, for a typical example, we have a circuit in which an 800 cycle e.m.f. E is acting through 1,000 ohms and we connect it to a load impedance of $100 + j0$ ohms—(see Fig. 22). In such a circuit, the current received would be

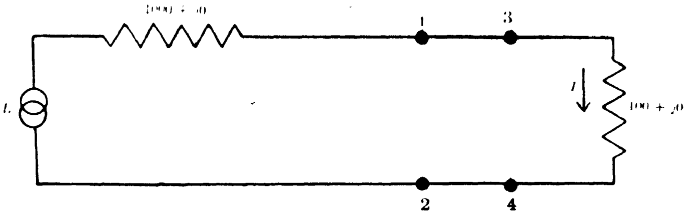


FIG. 22. Simple circuit in which a source of power is directly connected to an absorber of power.

$$I = \frac{E}{1{,}000 + 100} = .000909E \tag{81}$$

If now an ideal transformer of the ordinary type and of the proper impedance ratio (that is, $Z_P/Z_S = 1{,}000/100$) were inserted between the terminals 1–2 and 3–4, as shown in Fig. 23, the received current would be, from (9),

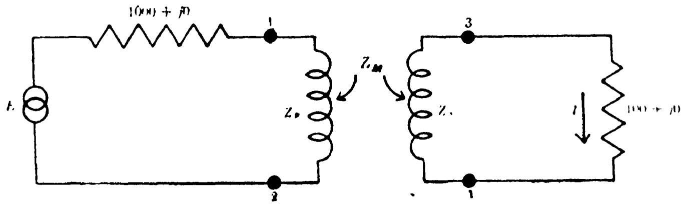
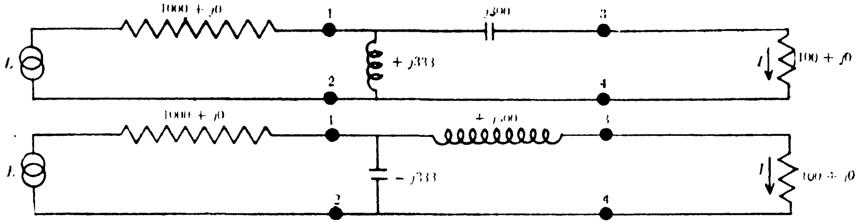


FIG. 23. Circuit in which a source of power is connected to an absorber of power by means of a two-winding transformer.

$$I = \frac{E}{2\sqrt{1{,}000 \times 100}} = .001581E \tag{82}$$

Assume now the equivalent two-reactance structure of an ideal transformer (see Fig. 21). Since we have $R_1 = 1{,}000$ and $R_2 = 100$, our circuit reduces to one of those shown in Fig. 24A or 24B. In either of



FIGS. 24A and 24B. Circuits in which a source of power is connected to an absorber of power by means of L networks composed of pure reactances.

these circuits the received current can be shown to be (by applying the formula for this type of circuit such as equation (3) of Appendix E)

$$I = .001581E \,\underline{/71^\circ 34'} \qquad \text{or} \qquad I = .001581E \,\overline{/71^\circ 34'} \qquad (83)$$

the sign of the angle depending upon which sign is taken for the reactances, or whether the circuit in Fig. 24A or 24B is assumed. Thus, if we insert between terminals 1–2 and 3–4 a two-reactance structure having the constants shown in Fig. 25, the received current—at 800 cycles—will

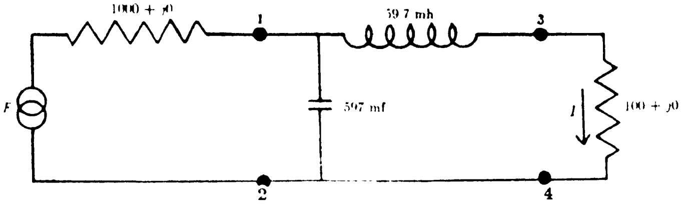


FIG. 25. Circuit in which a source of power is connected to an absorber of power by means of an L network comprised of a capacity and an inductance.

be $.001581E \,/71^\circ 34'$ or will be exactly the same in absolute magnitude as if an ideal transformer of the ordinary type—that is, having two inductive windings with mutual inductance between them—and of the optimum ratio were inserted between the terminals 1–2 and 3–4.

7.4 Practical Applications of Transformer and Transition Losses.

7.41 Transformers.—In order to get a clearer idea of what is meant by transformer and transition losses as well as to get a more concrete idea of their order of magnitude, let us consider the case of an e.m.f. E acting through a sending end impedance Z_1 of $240 - j123$ ohms and connected to a receiving end impedance Z_2 of $623 - j350$ ohms.

If now we connect these terminal impedances by an induction coil, of the type used in common battery subscribers' sets and having the 800 cycle constants shown in Fig. 26, and if we assume an electromotive force E acting in one of the impedances Z_1, the absolute value of the current in the other impedance Z_2 is $|I| = .001232E$ (see equation) (3)

With an ideal transducer connecting these two impedances, the magnitude of the current would be $.001293E$ (see equation (9)). If the impedances were directly connected together without any transformer, the corresponding current would be $.001016E$, while if they were connected together by an ideal transformer, the current would be $.001139E$ (see equation (8)).

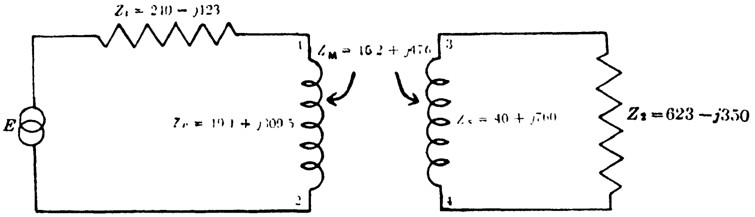


FIG. 26. Typical common battery induction coil connected between terminal impedances Z_1 and Z_2.

From these currents, and assuming a frequency of 796 cycles, it will be found that under the above circuit conditions, (1) the introduction of the induction coil has improved the efficiency of the circuit by 1.76 miles of standard cable, (2) the coil is .71 mile *above* the ideal transformer in efficiency so that the *transformer loss* at the junction of the two given terminal impedances—before the introduction of the induction coil—was $1.76 - .71 = 1.05$ miles. Likewise, the introduction of the coil removed all but .44 mile of the transition loss, i.e., the induction coil is only .44 mile below the ideal transducer in efficiency. Hence, the transition loss at the junction of the terminal impedances was approximately $1.76 + .44 = 2.20$ miles of standard cable. These figures are somewhat analogous to those obtained in Sect. 7.21 for a local battery type of induction coil when working between the best possible values for the terminating impedances and indicate how remarkably efficient a relatively cheap type of telephone transformer can be—particularly if it is working between terminal impedances having negative phase angles.

7.42 Transmitter Testing.—Fig. 27 shows schematically one of the circuits frequently used for testing the relative volume efficiency of two

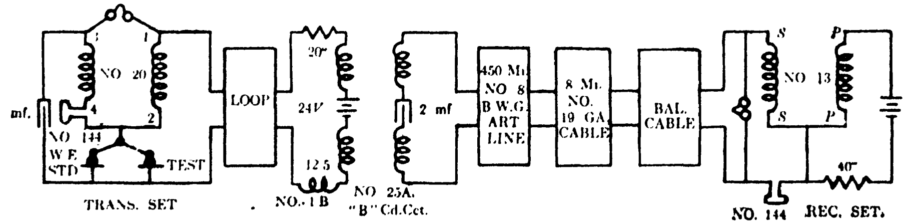


FIG. 27. Typical circuit for testing common battery transmitters.

transmitters. For equal agitations of the two transmitters, balancing cable is inserted until equal volumes of sound are obtained from each transmitter in the receiver of the receiving set. The number of miles of balancing cable required is taken as a measure of the relative volume efficiencies of the two transmitters in this circuit. It is to be noted that eight miles of standard cable are permanently connected in the circuit in order to eliminate such interaction or transition effects as have already been noted in Sect. 7.22 in connection with the reference to the *system reference standard.* This is done to insure that a given number of miles of balancing cable will produce a loss in the system which is proportional to the length of inserted cable.

The results of tests in such a circuit as this give relative volume efficiencies which are true only for this particular circuit, due to the fact that, in general, there is a different transition loss for each transmitter at the junction of the transmitter and the rest of the circuit. Thus, two transmitters of widely different resistances may show a *relative efficiency* in this circuit quite different from their relative efficiency when tested in some other circuit. In this particular circuit the 800 cycle impedance at the transmitter terminals is approximately $307 - j74$ ohms. It is, therefore, seen that with present commercial types of transmitters, ranging between 30 and 150 ohms in resistance, this circuit will discriminate in favor of the transmitter having the higher resistance.

Fundamentally, we are interested in the power delivered by a transmitter when it is working under its ideal condition; that is, when it is delivering power to an impedance that is the conjugate of its own internal impedance. The efficiency of one transmitter under this condition as compared to another transmitter under its ideal condition may be obtained from tests in the above testing circuit by calculating the transition losses existing at the junction of each transmitter and the rest of the circuit and by adding the difference of such losses to the relative or observed efficiencies of the instruments. That is, if L is the observed loss of transmitter T_1, as compared to transmitter T_2, and if L_1 is the transition loss for T_1, and L_2 is the transition loss for T_2, then $L_1 - L_2$ may be called the *correction loss* or L_C. The true *relative power efficiency* may then be obtained from $L + L_C$. It has been customary to determine the transition loss for 800 and 1,500 cycles and to take the arithmetic average of the results so obtained as the proper value.

From these corrected or relative power efficiency values we can again determine the relative efficiencies in any other circuit by reversing the above steps and adding to the relative power efficiency the difference in transition losses for the new circuit which will give us the relative or observed efficiency of the transmitters in that circuit. The above corrections assume that the direct current supplied to the transmitters is the same under the several conditions. If, however, the direct current varies, it is necessary to know the relative efficiency of the transmitters for these currents and also to have information on the change of transmitter resistance with direct current.

Two forms of transmitter testing circuits which are frequently used are those shown in Figs. 28A and 28B. The circuit shown in Fig. 28A makes it possible, by varying the taps on the transformer, to have both the standard and the test transmitter always working into approximately their own impedances. Consequently, the circuit makes it possible to determine readily the relative power efficiency of any two transmitters. The circuit shown in Fig. 28B accomplishes a similar purpose without the use of a transformer and its resultant frequency distortion.

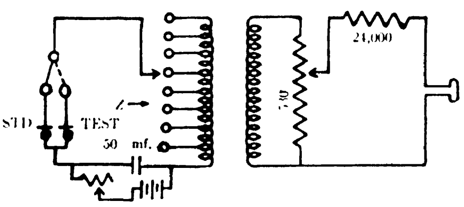


Fig 28A

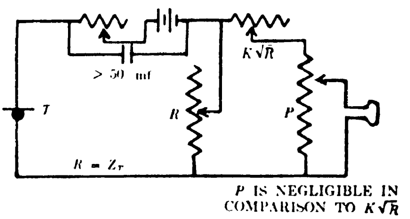


Fig 28B

FIGS. 28A and 28B. Power efficiency testing circuits for transmitters. Fig. A—Employing transformer coupling. Fig. B—Employing resistance coupling.

As an illustration of the use of transition corrections in transmitter efficiency analysis, let us consider the case in which two transmitters are tested in the circuit of Fig. 27. Assume that the impedance Z_1 of one transmitter is 30 ohms, that the impedance of the second transmitter Z_2 is 60 ohms and that by test the first transmitter T_1 is 2 miles less efficient than the second transmitter T_2. As previously stated, the impedance at the transmitter terminals of the testing circuit is $307 - j74$ ohms, or is $316 \,\underline{/13°.5}$ ohms at 800 cycles. Then when T_1 is used, the ratio of the impedances at the junction of the transmitter and the rest of the circuit is

$$316 \,\overline{/13°.5}/30 \,\underline{/0°} = 10.5 \,\overline{/13°.5}$$

From (16) the transformer loss corresponding to the above impedance ratio is found to be 5.4 miles. Similarly, from (21) the *phase difference loss* or the loss which must be added to the *transformer loss* in order to obtain the *transition loss* is found to be .08 mile. The transition loss at this junction is, therefore, 5.48 miles.

Similarly, at the junction of T_2 and the rest of the testing circuit, the impedance ratio is

$$\frac{316\,/\overline{13^\circ.5}}{60\,/\underline{0^\circ}} = 5.27\,/\overline{13^\circ.5}$$

which corresponds to a transformer loss of 3.20 miles, to a phase difference loss of .08 mile, as before, or to a transition loss of 3.28 miles. The difference between the two transition losses is seen to be 2.20 miles. Obviously, the circuit discriminates against T_1 which would, therefore, if working into its own impedance, be relatively more efficient than it appears to be by such a test. As shown above, it will be better by 5.48 miles while T_2 would only be better by 3.28 miles. Therefore, T_2 has the advantage over T_1 of 2.20 miles as noted above and the relative power efficiency of T_1 compared to T_2 is then $2.20 - 2.00 = +\ .20$ mile. That is, T_1 is inherently .20 mile more efficient than T_2.

From this corrected efficiency let us now consider what would be the relative efficiency of these two transmitters if they were working into a circuit having an impedance of $15\,/\underline{0^\circ}$. For T_1 the impedance ratio at the junction of the transmitter and the testing circuit would be $30\,/\underline{0^\circ}\,/\,15\,/\underline{0^\circ} = 2\,/\underline{0^\circ}$. From Figs. 2 and 3, or from formulae 16 and 21, this ratio corresponds to a transformer loss of .55 mile and to a zero phase difference loss—or to a transition loss of .55 mile.

Similarly, for T_2 the impedance ratio at the junction of the transmitter and the testing circuit will be $60\,/\underline{0^\circ}/15\,/\underline{0^\circ} = 4\,/\underline{0^\circ}$. This ratio corresponds to a transformer loss of 2.05 miles, a zero phase difference loss or a transition loss of 2.05 miles. The difference between the transition losses is, therefore, 1.50 miles and the circuit favors the low resistance transmitter. Hence, the 30 ohm transmitter will appear to be 1.50 miles better than its true relative efficiency and the observed relative efficiency on the 15 ohm impedance circuit will be $.20 + 1.50 = 1.70$ miles, which is the efficiency of T_1 compared to T_2 on this particular circuit.

From the nature of the phase difference loss it follows that for transmitters whose impedances have the same phase angles (such as carbon button transmitters whose impedances are essentially pure resistances) the phase difference losses will always be the same. Hence, it is not necessary actually to determine the phase difference loss—or even the transition loss—in those cases in which we are interested only in determining the *relative* efficiency of two instruments whose impedances have the same phase angle. It is only necessary to determine the transformer loss as indicated in the above example.

7.43 Receiver Testing.—Fig. 29 shows a cable testing circuit for measuring the relative efficiency of any two receivers. The conditions of test are similar to those for the transmitter testing circuit of Fig. 27. In most tests of standard receivers it is not necessary to consider d-c. supply as the receivers ordinarily employ permanent magnets. We do not, therefore, have to consider a variable impedance for the receiver as we do in the case of the transmitter. Corrections for relative efficiency are otherwise made in the same manner as already described for transmitters.

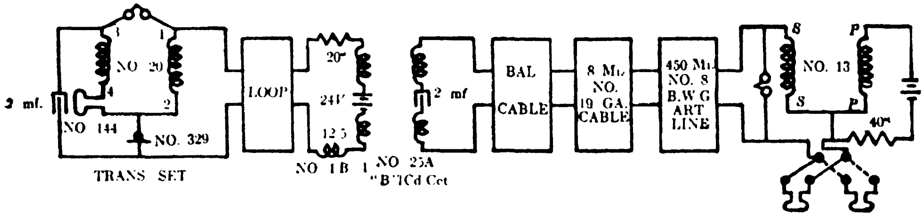


FIG. 29. Typical circuit for testing local and common battery receivers.

For convenience, the efficiency of any receiver under test is ordinarily referred to that of a *receiver standard*—the code number of the receiver which has most frequently been used for a receiver standard in the Bell System being designated as the No. 122. As a result, the *inherent efficiency* of any receiver is determined by *the ratio of the electrical power consumed by the receiver to the electrical power consumed by the No.* 122 *receiver standard when both receivers are delivering the same sound output.* This efficiency may, of course, be expressed in terms of miles of standard cable, napiers, or transmission units.

If we are dealing with an electromagnet receiver, it is necessary to consider its efficiency with various values of direct current or ampere-turns and the above testing circuit is then modified to vary the amount of direct current supplied to the receiver.

Another circuit which is frequently used for determining the inherent efficiency of receivers is the so-called *power efficiency testing circuit* shown in Fig. 30.

7.5 Terminal Impedances Which Result in no Reflection Losses.

7.51 Image Impedances of a Network.—It is frequently of interest to determine the values of the impedances, Z_{I_1} and Z_{I_2}—called, as we shall see later, the *image impedances* of the network—which, when terminating any structure, will result in no *reflection losses* at the junctions of the structure and the terminal impedances. No reflection loss

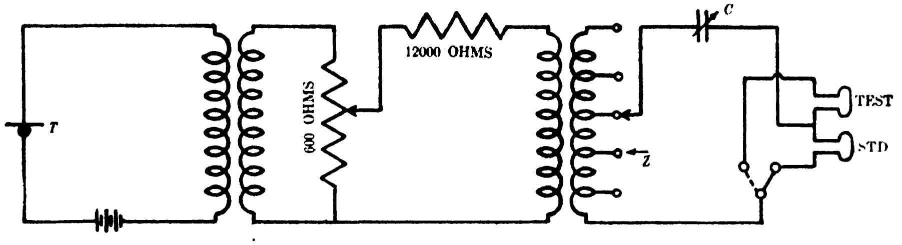


FIG. 30. Power efficiency testing circuit with transformer coupling, for testing receivers.

will exist (see Sect. 11.7) at any junction provided the impedance in one direction is equal to that in the other direction. Consequently, in Fig. 31

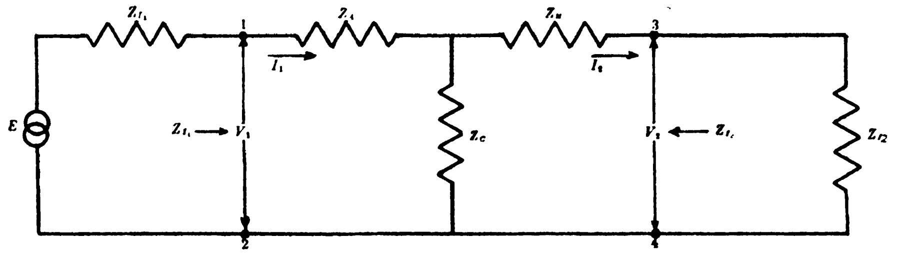


FIG. 31. T network terminated in impedances of such values that no reflection losses exist either at junction 1–2 or 3–4.

there will be no reflection losses at either the junction 1–2 or at 3–4 provided the impedances are, as indicated, the same in each direction. This is evidently equivalent to stating that

$$Z_{I_1} = Z_A + \frac{Z_C(Z_B + Z_{I_2})}{Z_C + Z_B + Z_{I_2}} \tag{84}$$

and

$$Z_{I_2} = Z_B + \frac{Z_C(Z_A + Z_{I_1})}{Z_C + Z_A + Z_{I_1}} \tag{85}$$

Solving these equations for Z_{I_1} and Z_{I_2}

$$Z_{I_1} = \sqrt{\frac{Z_A + Z_C}{Z_B + Z_C}(Z_AZ_B + Z_AZ_C + Z_BZ_C)} \tag{86}$$

and

$$Z_{I_2} = \sqrt{\frac{Z_B + Z_C}{Z_A + Z_C}(Z_AZ_B + Z_AZ_C + Z_BZ_C)} \tag{87}$$

If we let Z_O and Z_S be the open- and the short-circuit impedances respectively of the T network, as measured from the 1–2 terminals, and Z_O' and Z_S' be the corresponding impedances, as measured from the 34–

terminals, then

$$Z_O = Z_A + Z_C \tag{88}$$

$$Z_S = \frac{Z_AZ_B + Z_AZ_C + Z_BZ_C}{Z_B + Z_C} \tag{89}$$

$$Z_O' = Z_B + Z_C \tag{90}$$

$$Z_S' = \frac{Z_AZ_B + Z_BZ_C + Z_AZ_C}{Z_A + Z_C} \tag{91}$$

and equations (86) and (87) become

$$Z_{I_1} = \sqrt{Z_SZ_O} \tag{92}$$

$$Z_{I_2} = \sqrt{Z_S'Z_O'} \tag{93}$$

Equations (92) and (93) furnish a very simple means for determining the values of the terminal or *image impedances* Z_{I_1} and Z_{I_2}, in terms of the open- and short-circuit impedances of the given structure or network, which will result in no reflection loss at either junction of the structure and the terminating impedances. Consequently, since *the image impedances, Z_{I_1} and Z_{I_2} of any network, having two input and two output terminals, are defined as those impedances which will terminate the network in such a way that at either junction the impedance in either direction is the same, then such a definition is equivalent to saying that the image impedance at either end of a structure is the geometric mean of the open- and short-circuit impedance of the structure as determined from that end.* This is what Kennelly has called the *surge impedance.*

7.52 Image Transfer Constant of a Network.—*The image transfer constant** *of any passive network may be defined as one-half the natural logarithm of the vector ratio of the steady state volt-amperes entering and leaving the network when the latter is terminated in its image impedances. The ratio is determined by dividing the value of the volt-amperes at the point nearer the transmitting end by the value of the volt-amperes at the point more remote.*

With this definition we will prove that the image transfer constant of any structure, which will hereafter be called Θ, is also equal to the hyperbolic anti-tangent of the square root of the ratio of the short-circuit impedance to that of the open-circuit impedance of the structure. In other words, referring to Fig. 31, it will be proved that

$$\Theta \equiv \frac{1}{2}\log_e \frac{V_1I_1}{V_2I_2} = \tanh^{-1}\sqrt{\frac{Z_S}{Z_O}} = \tanh^{-1}\sqrt{\frac{Z_S'}{Z_O'}} \tag{94}$$

---

*O. J. Zobel has suggested calling this term simply the *transfer constant.* The omission of the word *image* would certainly simplify the expression and might not lead to any confusion.

By Kirchhoff's Laws it can be shown that

$$\frac{I_1}{I_2} = \frac{Z_B + Z_C + Z_{I_2}}{Z_C} \tag{95}$$

and that

$$\frac{V_1}{V_2} = \frac{Z_A Z_B + Z_A Z_C + Z_B Z_C + Z_{I_2}(Z_A + Z_C)}{Z_C Z_{I_2}} \tag{96}$$

Also from (88) and (89) or from (90) and (91):

$$\sqrt{\frac{Z_S}{Z_O}} = \sqrt{\frac{Z_S'}{Z_O'}} = \sqrt{\frac{Z_A Z_B + Z_A Z_C + Z_B Z_C}{(Z_A + Z_C)(Z_B + Z_C)}} \tag{97}$$

Hence, it is only necessary to prove that

$$\frac{1}{2}\log_\epsilon \frac{[Z_B + Z_C + Z_{I_2}][Z_A Z_B + Z_A Z_C + Z_B Z_C + Z_{I_2}(Z_A + Z_C)]}{Z_C^2 Z_{I_2}}$$

is equal to

$$\tanh^{-1}\sqrt{\frac{Z_S'}{Z_O'}} \qquad \text{or to} \qquad \tanh^{-1}\sqrt{\frac{Z_A Z_B + Z_B Z_C + Z_A Z_C}{(Z_A + Z_C)(Z_B + Z_C)}}$$

where Z_{I_2} has the value given by (93) or where

$$Z_{I_2} = \sqrt{Z_S' Z_O'} = \sqrt{\frac{Z_B + Z_C}{Z_A + Z_C}(Z_A Z_B + Z_B Z_C + Z_A Z_C)} \tag{98}$$

By employing the relation given in equation (53) of Appendix A, it can be shown that

$$\begin{aligned}\tanh^{-1}\sqrt{\frac{Z_S'}{Z_O'}} &= \frac{1}{2}\log_\epsilon \frac{1 + \sqrt{\dfrac{Z_S'}{Z_O'}}}{1 - \sqrt{\dfrac{Z_S'}{Z_O'}}} = \frac{1}{2}\log_\epsilon \frac{Z_O' + \sqrt{Z_O' Z_S'}}{Z_O' - \sqrt{Z_O' Z_S'}} \\ &= \frac{1}{2}\log_\epsilon \frac{Z_O' + Z_{I_2}}{Z_O' - Z_{I_2}} = \frac{1}{2}\log_\epsilon \frac{Z_B + Z_C + Z_{I_2}}{Z_B + Z_C - Z_{I_2}}\end{aligned} \tag{99}$$

But

$$\begin{aligned}&\frac{Z_B + Z_C + Z_{I_2}}{Z_B + Z_C - Z_{I_2}} \\ &\quad = \frac{[Z_B + Z_C + Z_{I_2}][Z_A Z_B + Z_A Z_C + Z_B Z_C + Z_{I_2}(Z_A + Z_C)]}{Z_C^2 Z_{I_2}}\end{aligned} \tag{100}$$

Hence

$$\frac{1}{2}\log_e \frac{V_1 I_1}{V_2 I_2} = \tanh^{-1}\sqrt{\frac{Z_S}{Z_O}} = \tanh^{-1}\sqrt{\frac{Z_S'}{Z_O'}} \tag{101}$$

and the relation expressed in (94) is thus proved to be valid. We can, therefore, readily express the image transfer constant Θ of any network in terms of its open- and short-circuit impedances as well as in terms of the volt-amperes entering and leaving the network—when the latter is terminated in its image impedances, Z_{I_1} and Z_{I_2}. This matter is discussed further in Sect. 11.22.

CHAPTER VIII

THÉVENIN'S THEOREM AND EQUIVALENT (*T*) NETWORKS

8.0 Thévenin's Theorem.—Practically all of the formulae which have so far been derived, such as, for example, those for transformer and transition losses, have been derived on the assumption that we have had simple series circuits, that is, an e.m.f. E acting in series with an impedance and terminating at the distant end in a simple receiving impedance. As a matter of fact even a line which has no actual apparatus bridged across it has shunt capacitance and leakance so that actually every line is made up of both series and shunt impedances. In order then to show that the above-mentioned formulae are of general use and can be made to apply to any circuit, no matter whether it be of the simple series type such as has already been discussed, or whether it be of the most complex type imaginable, let us consider a simple but extremely useful proposition which is called *Thévenin's Theorem.*

8.1 This very useful and important theorem* may be stated as follows. If a source of simple harmonic electromotive force E, and of internal impedance Z_T, is connected to the input terminals (1–2) of any invariable network and if an impedance Z_R is connected to the output terminals (3–4) of the same network, the current in Z_R will be $E'/(Z_T' + Z_R)$, where E' is the open-circuit voltage across the terminals 3–4 and Z_T' is the impedance that would be measured across 3–4 with the source replaced by a simple series impedance equal to its internal impedance Z_T. This is the same as saying that in so far as the terminals 3–4 are concerned the combination of source E and four-terminal network, 1–2 and 3–4, is equivalent to a new source of electromotive force E' and a *series* impedance Z_T', where E' and Z_T' are independent of the impedance Z_R externally connected across the 3–4 terminals.

Thévenin's Theorem follows at once—as may be seen from the fact that the received currents I_R in the circuits shown in Figs. 1A and 1B are identical—from the important and well-known proposition that any passive network, comprised of invariable elements and having two input terminals 1–2 and two output terminals 3–4, is externally equivalent, at the frequency of the source of e.m.f., to a properly designed simple three-parameter network such as a T or Π network. The general proof of this[*] is somewhat involved but the fact that any specific passive network can, at any single frequency, be reduced mathematically to a single equivalent T network becomes almost self-evident when it is considered that any two T networks (as shown in Fig. 2A) can be combined into a single equivalent T network (see Fig. 2B).

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* See *Comptes Rendus* for 1883, v. 97, p. 159. Also J. B. Pomey's "Cours d'Électricité Théorique," v. 1, p. 136, and A. Vaschy's "Traité d'Électricité et De Magnetisme," v. 1, p. 153.

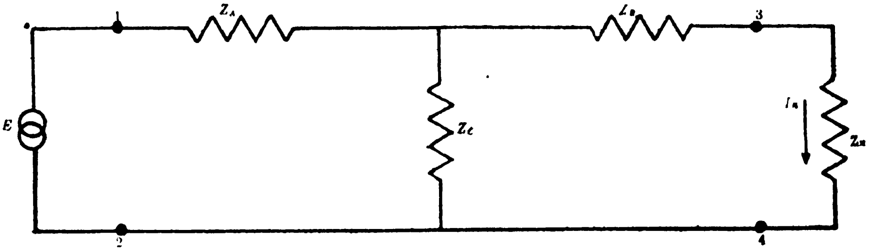

FIG. 1A.

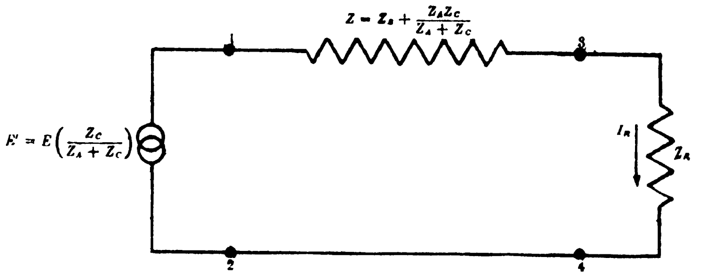
$Z = Z_B + \frac{Z_A Z_C}{Z_A + Z_C}$
1
3
$E' = E\left(\frac{Z_C}{Z_A + Z_C}\right)$
I_R
Z_R
2
4

FIG. 1B.

FIGS. 1A and 1B. Equivalent circuits—insofar as the received current I_R is concerned. Fig. A—Circuit employing a T network which may represent any structure. Fig. B—Equivalent circuit employing a simple series impedance.

Consider for example the two circuits shown in Figs. 2A and 2B. The circuit shown in Fig. 2B is said to be the *equivalent* of that in Fig. 2A if for any values of Z_L, Z_R and E the current I_L entering the terminals 1–2 in Fig. 2A is equal in phase and in magnitude to the current I_L' entering the terminals 1–2 in Fig. 2B and if the current I_R leaving the terminals 3–4 in Fig. 2A is equal to the current I_R' leaving the terminals 3–4 in Fig. 2B. By applying Kirchhoff's Laws to each of these circuits, it can be shown that the expression for I_L is identical with that for I_L' and that the expression for I_R is identical with that for I_R'; hence, the two T networks (Z_A, Z_B, Z_C and Z_D, Z_E, Z_F) shown in Fig. 2A are said to be equivalent to the single T network shown in Fig. 2B.

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[*] See "Cisoidal Oscillations" by G. A. Campbell, *Transactions of the A. I. E. E.* (1911), Volume XXX, Part II, pp. 873–909. "Theory of Variable Dynamical Electrical Systems," by H. W. Nichols, *Physical Review*, 1917, Vol. 10, pp. 171–193.

It is evident from the above that if we have any sort of a complex line or circuit made up of series and shunt impedances, we can combine them one by one until we have the entire circuit reduced to a single equivalent *T* network.

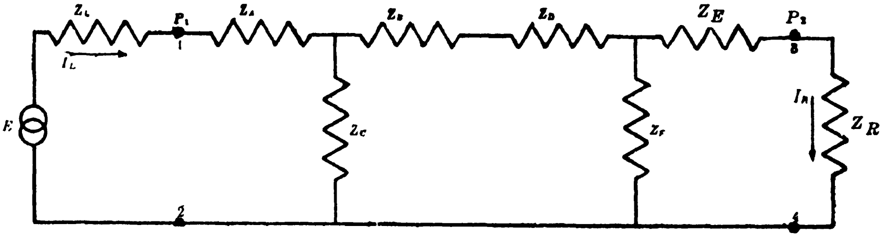


FIG. 2A.

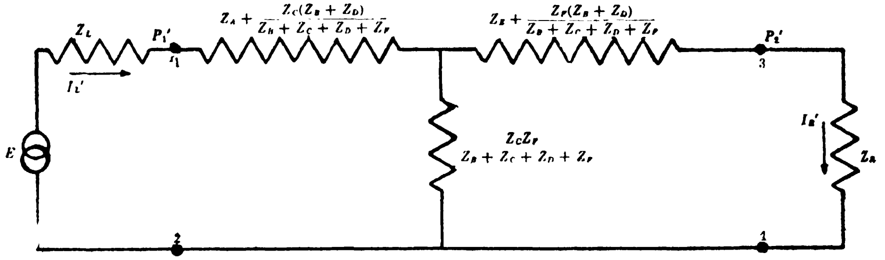

Z_L
P_1'
$Z_A + \frac{Z_C(Z_B + Z_D)}{Z_B + Z_C + Z_D + Z_F}$
$Z_E + \frac{Z_F(Z_B + Z_D)}{Z_B + Z_C + Z_D + Z_F}$
P_2'
1
3
I_L'
E
$\frac{Z_C Z_F}{Z_B + Z_C + Z_D + Z_F}$
I_R'
Z_R
2
4

FIG. 2B.

FIGS. 2A and 2B. Circuits showing the external equivalence of two *T* networks and a single *T* network. Fig. A—Circuit employing two *T* networks. Fig. B—Circuit employing an equivalent single *T* network.

Having pointed out that any line, no matter how complex, is reducible at any frequency to a single *T* network, let us consider a little more in detail the practical importance of *Thévenin's Theorem*. We now know that any composite circuit, such as is shown in Fig. 3, is exactly equivalent, in so far as I_O and I_R are concerned, to that shown in Fig. 4 in which Z_Q, Z_R and Z_S, as well as Z_X, Z_Y and Z_Z, can be determined provided the constants of all the component parts of the actual circuit are known.

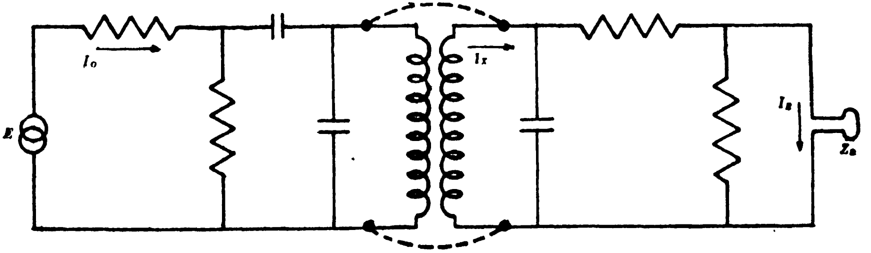


FIG. 3. Diagram representing two complicated circuits connected by a transformer or other device at the point *P*

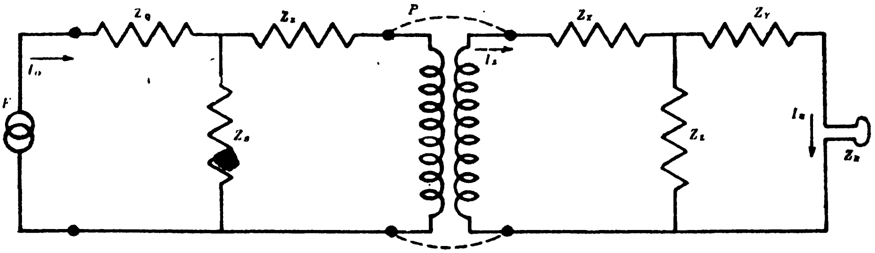


FIG. 4. Circuit employing two T networks but equivalent, in so far as I_O and I_R are concerned, to the circuit shown in Fig. 3.

In view of *Thévenin's Theorem,* the above circuit can then be reduced, in so far as the currents I_O and I_X are concerned, to the simple series circuit which is shown in Fig. 5. In this circuit, we can at once calculate

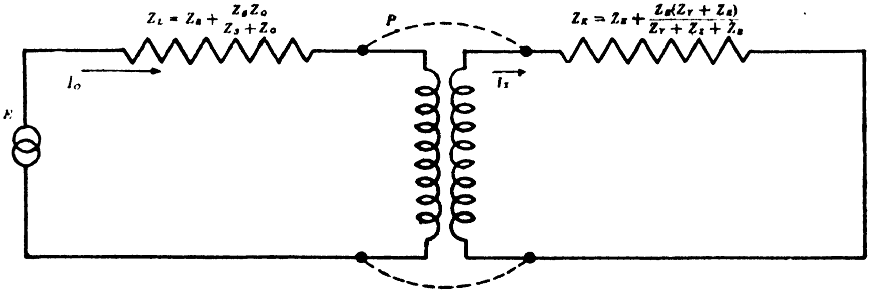

$Z_L = Z_x + \frac{Z_sZ_o}{Z_s + Z_o}$
P
$Z_K = Z_x + \frac{Z_t(Z_Y + Z_R)}{Z_Y + Z_t + Z_R}$
I_o
E
I_x

FIG. 5. Circuit, employing simple series impedances, in which the transmission loss caused by the introduction of any device at the point P is identical to the loss caused by the introduction of the same device in Fig. 3 or Fig. 4.

I_X in terms of E'. Now, if in the actual circuit we change nothing to the right of the point P, I_X will evidently always bear a constant ratio to I_R. Consequently, if E' remains constant—as it must if the electromotive force E and the network to the left of point P remain constant—and we compute the values of I_X with the line at P connected together first by one kind of apparatus and then by another, the ratio of I_X under these two conditions will be the same as the ratio of I_R under the same conditions. Hence, for the purpose of computing the loss or gain due to the insertion of any apparatus at the point P, it is only necessary to know what are the values of the impedances, Z_L and Z_K, looking in each direction from the point at which the apparatus is to be inserted and treat them as series impedances with a constant electromotive force acting through one of them. *If, therefore, any apparatus is inserted in a circuit, no matter how complex the circuit may be, we can at once tell what the transmission loss or gain due to such insertion will be, provided we simply know the impedances in each direction from the point under consideration.*

The above fact makes it possible for one to derive formulae on the assumption of simple series impedances and to apply them, by a proper interpretation, to any type of telephone circuit, no matter how complex its structure. For example, since the circuit shown above is equivalent to an electromotive force E' acting through the series impedance Z_L, it is evident, from what has been proved before regarding the simple series circuit, that *the greatest possible power is absorbed from any given circuit when the latter is terminated in a receiving end impedance which is the conjugate of the impedance looking towards the electromotive force and this power is the square of the e.m.f. divided by four times the effective resistance of the source of power.*

The above proof also shows that the various formulae which were derived for transition losses, transformer losses, etc., on the assumption of simple series circuits, are general in their application provided that in their use Z_T and Z_R are considered to be the impedances measured in the two directions looking away from the point at which the transformer or other apparatus is to be inserted.

In this connection it is interesting to note, from the curves in Fig. 2 of Chapter VII, the relatively small gains which can be obtained by inserting an ideal transformer, of the optimum ratio, in a circuit—provided the impedance in one direction is not materially greater than it is in the other direction. For example, if the impedances have the same phase angles and the impedance in one direction is 50 per cent greater than it is in the other direction, the insertion of an ideal transformer will only give a gain of .2 mile of standard cable. Similarly, if the impedance in one direction is twice that which it is in the other direction, the gain to be obtained by the insertion of an ideal transformer at that junction is not much greater than .5 mile. Inasmuch as the usual type of telephone transformer has losses, due to the d-c. resistance of its windings, etc., of the order of several tenths of a mile, it is evident that there is, in general, nothing much to be gained—in increased efficiency—by using a transformer unless the impedance ratio at the junction at which it is to be inserted is at least as great as 2 : 1.

8.2 Equivalent *T* Network of a Transformer.—The use of equivalent T or Π networks is often of great value in making it possible to analyze the transmission losses introduced by certain apparatus much more readily than could otherwise be done. For example, the use of an equivalent T network to represent a two-winding transformer makes it relatively easy to see the effect of changes in the transformer constants upon the transmission efficiency of the circuit. The equivalent *T* network of such a transformer is shown in Fig. 23B of Appendix D.

If the windings of the transformer shown in Fig. 23A of Appendix D are in such a direction that, when terminals 2 and 3 are connected together, they are in a series aiding connection, the corresponding equivalent *T* network has for its series arms $Z_P - Z_M$ and $Z_S - Z_M$, respectively, and for its shunt arm Z_M. The other signs shown in Fig. 23B hold when the direction of the windings is such that connecting terminals 2 and 3 together gives a series opposing connection. The *T* network in which the series arms consist, respectively, of the differences of the primary and the mutual impedances, and of the secondary and the mutual impedances is the most convenient form for ordinary considerations and will, therefore, be briefly discussed.

If, to simplify matters, we consider a unity ratio transformer (that is, one in which $Z_S = Z_P$), the impedance $(Z_P - Z_M)$ of each of the series arms of the equivalent *T* network may be regarded as containing one-half of the total d-c. resistance of the transformer and one-half of the total leakage reactance. By means of the methods outlined in Chapter IX it is then possible to compute how much transmission loss is introduced by the series arms—or by the d-c. resistance and the leakage reactance. In a like way it is possible to determine the loss caused by the shunt arm or by the mutual impedance Z_M. In most well-designed transformers the transmission loss caused by the two series arms should be of the same order of magnitude as that caused by the bridged or shunt arm. This statement is analogous to that frequently made in power transformer design work, namely, that for maximum efficiency the copper losses should be approximately equal to the core losses.

The impedances of the series arms of the equivalent *T* network are, in an efficient transformer, relatively small, while that of the shunt arm is relatively large as compared with the circuit impedances between which the transformer is to be connected. Winding such a unity ratio transformer with a larger number of turns of smaller gauge wire will evidently increase the impedance of both the series and the shunt arms of the equivalent *T* network, thereby increasing the loss caused by the series impedances $(Z_P - Z_M)$ and decreasing the loss caused by the shunt or mutual impedance Z_M. It is thus seen that by a study of the relative magnitudes of the losses caused by the series and the shunt arms of the equivalent *T* network of a unity ratio transformer it is possible to determine whether or not there are too many or too few turns on the transformer for maximum efficiency.

In the case of a transformer having an impedance ratio that is not unity, that is, $Z_S \neq Z_P$, the equivalent T network of the transformer, as given in Fig. 23B of Appendix D, is not as useful as in the preceding case of a unity ratio transformer. This is due to the fact that in the equivalent T network of the transformer which does not have a unity ratio, the impedances of the two series arms $(Z_P - Z_M)$ and $(Z_S - Z_M)$ are not, in general, small quantities nor are they equal to each other either in magnitude or in phase.

Consequently, in analyzing the losses in such a transformer it is customary to regard it (as shown in Fig. 25B of Appendix D) as a combination of an equivalent T network of a unity ratio transformer and an ideal transformer of the optimum impedance ratio. The particular structure shown in Fig. 25B is strictly equivalent to the transformer shown in Fig. 25A only when the self impedances (Z_P and Z_S) of the latter have identically the same phase angle. This is, however, a condition that holds very closely in most all well-designed transformers.

If then we consider the 1–2 terminals of the equivalent T network to be connected to the normal impedance with which the 1–2 terminals of the transformer shown in Fig. 25A are to be connected and if the 5–6 terminals of the T network are connected to Z_P/Z_S of the impedance between which the 3–4 terminals of the transformer are to be connected, we have a means for analyzing the various losses occurring in such a transformer that is essentially similar to that holding for the unity ratio type of transformer.

For reference purposes, examples of various structures and their corresponding equivalent T or Π networks are given in Figs. 22 to 34 inclusive of Appendix D.

CHAPTER IX

THE COMPUTATION OF TRANSMISSION LOSSES

9.0 It is evident that Thévenin's Theorem, taken in conjunction with the fact that any circuit can be reduced at any frequency to a single *T* or Π network, enables us to simplify greatly all transmission problems which involve simply the determination of the *ratio* of the received current or energy, before and after making a change in the circuit.

If, as is seldom the case, it is required to determine the actual numerical value of the received current in any structure, in terms of a known electromotive force acting in the circuit and the various impedance elements of the circuit, we can, in general, do so either by using specific formulae applying to the circuit in question—such as those given in Appendix E—or by the somewhat more general method outlined below.

Fig. 1 represents any structure comprised of series and shunt impedances. In computing the transmission loss introduced by the insertion of such a structure between terminals 1–2 and 3–4, the quantities indicated below are determined in turn.

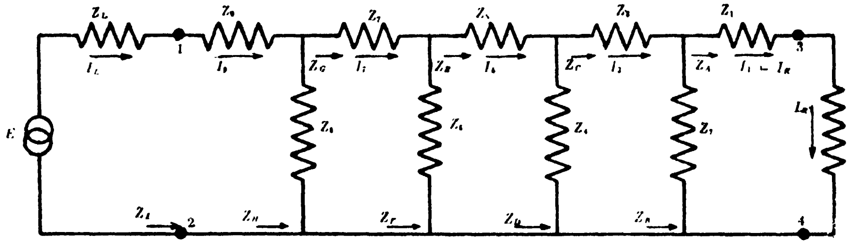


FIG. 1. Ladder type of structure whose transmission loss is to be determined.

$$Z_A = Z_1 + Z_R \quad (1) \qquad \frac{I_R}{I_3} = \frac{I_1}{I_3} = \frac{Z_2}{Z_2 + Z_A} \quad (2) \qquad Z_B = \frac{I_1}{I_3} Z_A \quad (3)$$

$$Z_C = Z_3 + Z_B \quad (4) \qquad \frac{I_3}{I_5} = \frac{Z_4}{Z_4 + Z_C} \quad (5) \qquad Z_D = \frac{I_3}{I_5} Z_C \quad (6)$$

$$Z_E = Z_5 + Z_D \quad (7) \qquad \frac{I_5}{I_7} = \frac{Z_6}{Z_6 + Z_E} \quad (8) \qquad Z_F = \frac{I_5}{I_7} Z_E \quad (9)$$

$$Z_G = Z_7 + Z_F \quad (10) \qquad \frac{I_7}{I_9} = \frac{I_7}{I_L} = \frac{Z_8}{Z_8 + Z_G} \quad (11) \qquad Z_H = \frac{I_7}{I_9} Z_G \quad (12)$$

$$Z_I = Z_9 + Z_H \quad (13)$$

Whence

$$I_L = I_9 = \frac{E}{Z_L + Z_I} \tag{14}$$

Then

$$I_R = \frac{I_R}{I_3} \times \frac{I_3}{I_5} \times \frac{I_5}{I_7} \times \frac{I_7}{I_9} \times I_9 \tag{15}$$

The factors necessary to get I_R in (15) are directly obtainable from (2), (5), (8), (11), and (14).

Without the structure in the circuit, the current (I_R') flowing through Z_R would be

$$I_R' = \frac{E}{Z_L + Z_R} \tag{16}$$

The ratio of the currents given by (15) and (16) makes it possible to determine the transmission loss caused by the introduction of the structure. The loss, in miles of standard cable L, is then given by the relation

$$L = 21.13 \log_{10} \frac{I_R'}{I_R} \tag{17}$$

The method outlined above is particularly general in its application in view of the fact that, as previously stated, all structures, comprised of invariable elements, can be reduced, at any given frequency, to equivalent T networks. A means of calculating the transmission loss of some structures that is, in general, simpler than the method outlined above is by means of a knowledge of the iterative impedances and of the propagation constant or of the image impedances and the image transfer constant of the structure. This will be taken up in detail in Sect. 11.8.

In most cases, however, we do not need to know the absolute values of the alternating currents in a telephone circuit but simply their relative value. This is especially true in those cases where one is interested in determining the *transmission loss* caused by the introduction of any apparatus into a telephone circuit. *The transmission loss caused by any modification of a telephone system is the loss in the power delivered to the receiving device of the system and is measured by the ratio of the received powers before and after such a modification.*

Consequently, the problem of finding the transmission loss due to a piece of apparatus inserted in or bridged across a telephone line is the problem of finding how the current entering the receiving set is altered by the inserted or bridged apparatus. The apparatus may consist of series or bridged impedances, transformers, lengths of cable or open wire lines, etc.

In general, there are four terminals to a piece of inserted apparatus; that is, two connected to the telephone circuit going in one direction and two connected to the circuit going in the other direction. A simple bridge may be considered as having four terminals by including with it "a foot of line" on each side of it. The above statement of four terminals assumes the usual condition of the two sides of the circuit, and also of the inserted apparatus, being nearly symmetrical, so that the presence of the ground or other neighboring bodies need not be considered. If this condition does not hold, the apparatus has another terminal to be considered—the ground.

As stated in Sect. 1.1, it is often assumed, in rough work, that the transmission loss calculated at a frequency of about 1,100 cycles is perhaps as satisfactory a value as any, in estimating the transmission loss existing with voice currents. This is not a safe assumption, however, when the loss varies widely for different frequencies between 250 and 2,500 or perhaps 3,000 cycles, as is apt to be the case where losses due to the insertion of condensers or the omission of loading coils, etc., are being considered. As pointed out in Chapter I, frequencies outside of the limits given above are not particularly important in the transmission of commercial telephonic speech although frequencies as high as 8,000 cycles must oftentimes be considered in the transmission of high quality speech or music.

It is evident that computations or measurements at no single frequency are as good as at two or more frequencies because of the possibility of a sharp resonance at a frequency near that used, and the erroneous results which might be caused by such a resonance. At present, it is quite general in rough computation work to use an average of the results of computations obtained at 800 and 1,500 cycles. In certain types of problems, however, it may oftentimes be necessary to consider a larger number of frequencies and to use a weighted average in obtaining the final result.

We have shown by Thévenin's Theorem that if one is interested only in knowing the *ratio* of the current in a circuit before and after inserting any apparatus, it is simply necessary to take the line at the place where the apparatus is to be inserted, find the impedance of the line in both directions and then assume that a constant e.m.f. is acting *through* one impedance, thus causing a current to flow in it and in the second impedance, and then compute how the current flowing in the second impedance is altered by the insertion of the apparatus. The ratio of the currents with and without the apparatus will give the factor which in turn may be translated into miles of standard cable, napiers or TU by formulae such as are given in Chapter II. The only assumption made in the above is that the change produced in the magnitude of the currents by the inserted apparatus does not affect the constants of the circuit or the magnitude of the e.m.f. acting in the transmitter.

9.1 Transmission Losses Due to the Insertion of Series Impedance. —Any apparatus inserted in a telephone line may offer either series or bridged impedance to the line or it may offer both. Let us first consider the case of apparatus inserting simple series impedance in the line. Let the line at the point where the apparatus is to be inserted have an impedance Z_1 in one direction and an impedance Z_2 in the other direction. Then if we assume, in accordance with Thévenin's Theorem, an e.m.f. E acting through Z_1 on to Z_2, the current in Z_2 is $E/(Z_1 + Z_2)$. Now, if apparatus having a series impedance Z_C be inserted between Z_1 and Z_2, the current flowing into Z_2 is $E/(Z_1 + Z_2 + Z_C)$.

Whence, the absolute magnitude of the ratio r of the currents; after and before inserting the apparatus, is:

$$r = \frac{Z_1 + Z_2}{Z_1 + Z_2 + Z_C} \tag{18}$$

r may then be changed into loss, in terms of any of the various types of transmission units, by means of the formulae given in Chapter II.

9.2 Transmission Losses Due to the Insertion of Bridged Impedance. Suppose now that instead of the apparatus offering a series impedance to the circuit it had offered a bridged impedance Z_S. As before, the current in Z_2 before the insertion of the apparatus is $I_1 = E/(Z_1 + Z_2)$ (see Fig. 2A). After the insertion of the bridge (see Fig. 2B), it is

$$I_2 = \frac{EZ_S}{Z_1Z_S + Z_2Z_S + Z_1Z_2} \tag{19}$$

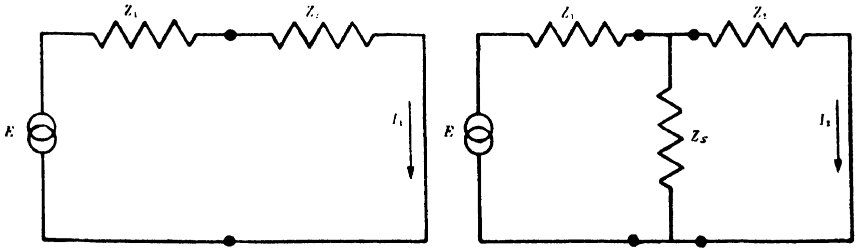


Fig. 2A Fig. 2B

FIGS. 2A and 2B. Circuits illustrating the method of calculating transmission losses due to a simple bridged impedance.

Then from the above expressions

$$r \equiv \left|\frac{I_2}{I_1}\right| = \left|\frac{1}{1 + \dfrac{Z_1Z_2}{Z_S(Z_1 + Z_2)}}\right| \tag{20}$$

Equating (18) and (20) gives

$$\left|\frac{Z_1 + Z_2}{Z_1 + Z_2 + Z_C}\right| = \left|\frac{1}{1 + \dfrac{Z_1Z_2}{Z_S(Z_1 + Z_2)}}\right| \tag{21}$$

or

$$Z_CZ_S = Z_1Z_2 \tag{22}$$

From this relation we may draw the following useful conclusion. *If any network is inserted in series between any two terminal impedances—whose product is a constant pure resistance—the transmission loss caused thereby will be exactly the same, at all frequencies, as if the corresponding inverse network of constant resistance product (see Sect.* 18.1) *is shunted across the circuit.*

An illustration of the use of this principle may be seen by referring to Figs. 3A and 3B. In other words, the network composed of the

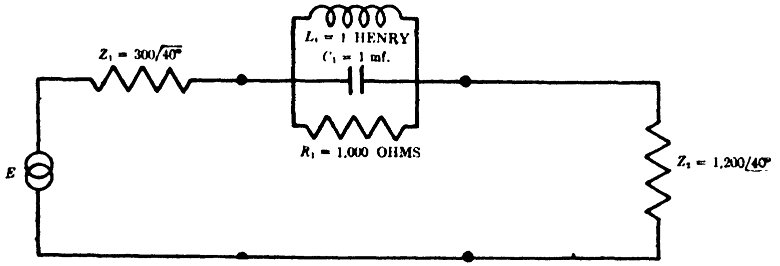


FIG. 3A.

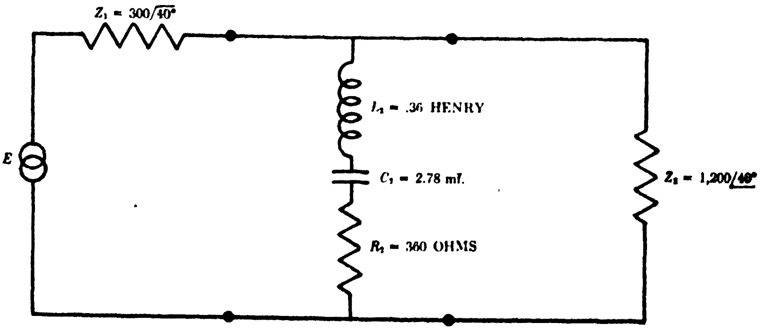


FIG. 3B.

FIGS. 3A and 3B. Typical circuits, (the product of whose terminal impedances is a *constant resistance*) whose efficiency is the same *at all frequencies*.

parallel elements $L_1 = 1$ henry, $C_1 = 1$ microfarad and $R_1 = 1{,}000$ ohms in Fig. 3A causes exactly the same loss *at all frequencies* as does the corresponding *inverse network* in Fig. 3B. The elements of the inverse network, made up of the series elements $L_2 = .36$ henry, $C_2 = 2.78$ microfarads and $R_2 = 360$ ohms, are related to those of the network in Fig. 3A by the relations given in equation (1) of Chapter XVIII or

$$L_2/C_1 = L_1/C_2 = R_1R_2 = Z_1Z_2 = 360{,}000 \text{ ohms}$$

9.3 Series and Bridged Impedance.—Let us now consider the last and general case in which the apparatus offers both series and bridged impedance to the circuit. Suppose the apparatus has a shunt impedance of Z_S and series impedances of Z_A and Z_B as shown in Fig. 4B.

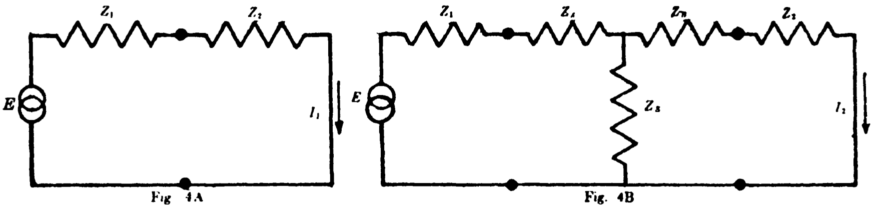


Fig 4A

Fig. 4B

FIGS. 4A and 4B. Circuits illustrating the method of calculating transmission losses due to the insertion of any T network.

As stated before, all apparatus—including lines, transformers, etc.—can be represented electrically by such an equivalent T network consisting of two series impedances and one shunt impedance. In this case

$$r \equiv \left|\frac{I_2}{I_1}\right| = \left|\frac{Z_S(Z_1 + Z_2)}{(Z_1 + Z_A)(Z_2 + Z_B + Z_S) + Z_S(Z_2 + Z_B)}\right| \tag{23}$$

If in any case r comes out larger than unity, the *loss* in such a case will actually be a *gain*. In other words, in such a case a greater current will flow into Z_2 with the apparatus inserted than when it is omitted and, consequently, there will be more power in the receiver after inserting the apparatus than there was before. It will also be noted that the loss depends simply upon the *absolute* or *numerical* value of the current ratio r and that the *phase difference* between I_1 and I_2 does not enter into the problem. This is due to the fact that, as found by Helmholtz,* the ear cannot detect phase differences when listening to incoming speech from a single receiver, as is usual in commercial telephony. In other words, the ear analyzes the complex wave into its constituents independently of their phase relations.

* "Sensations of Tones," by Von Helmholtz, Chapter VI.

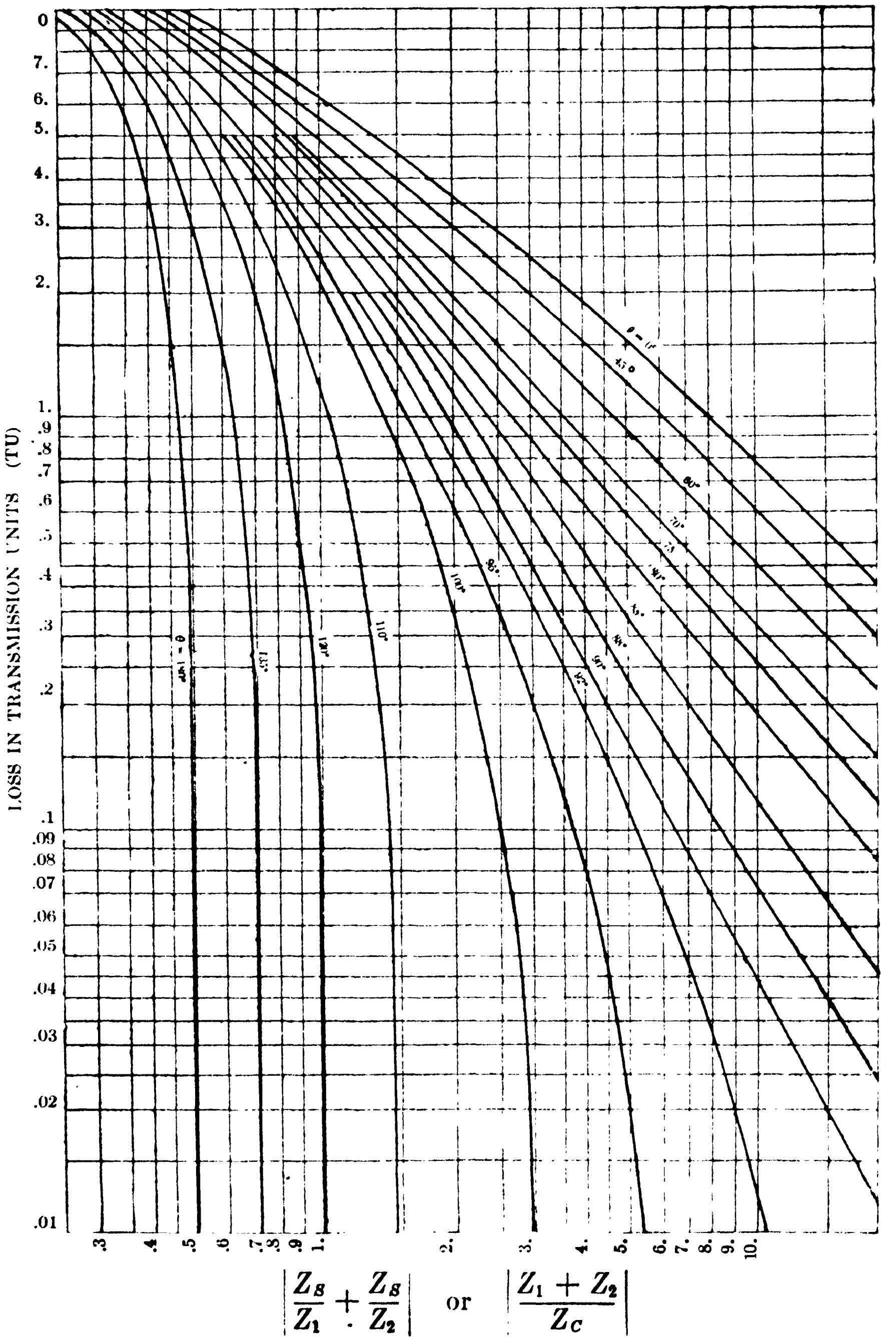
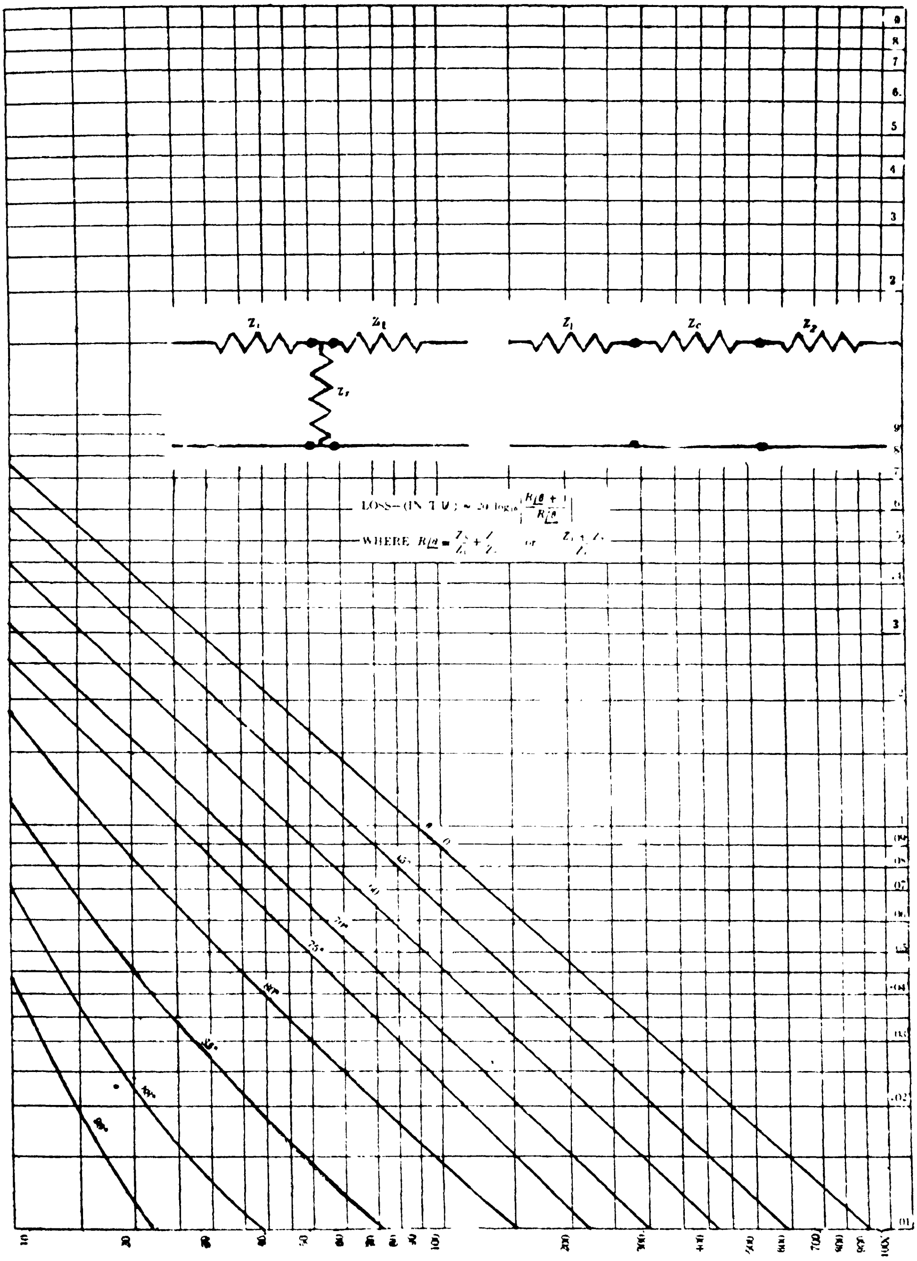
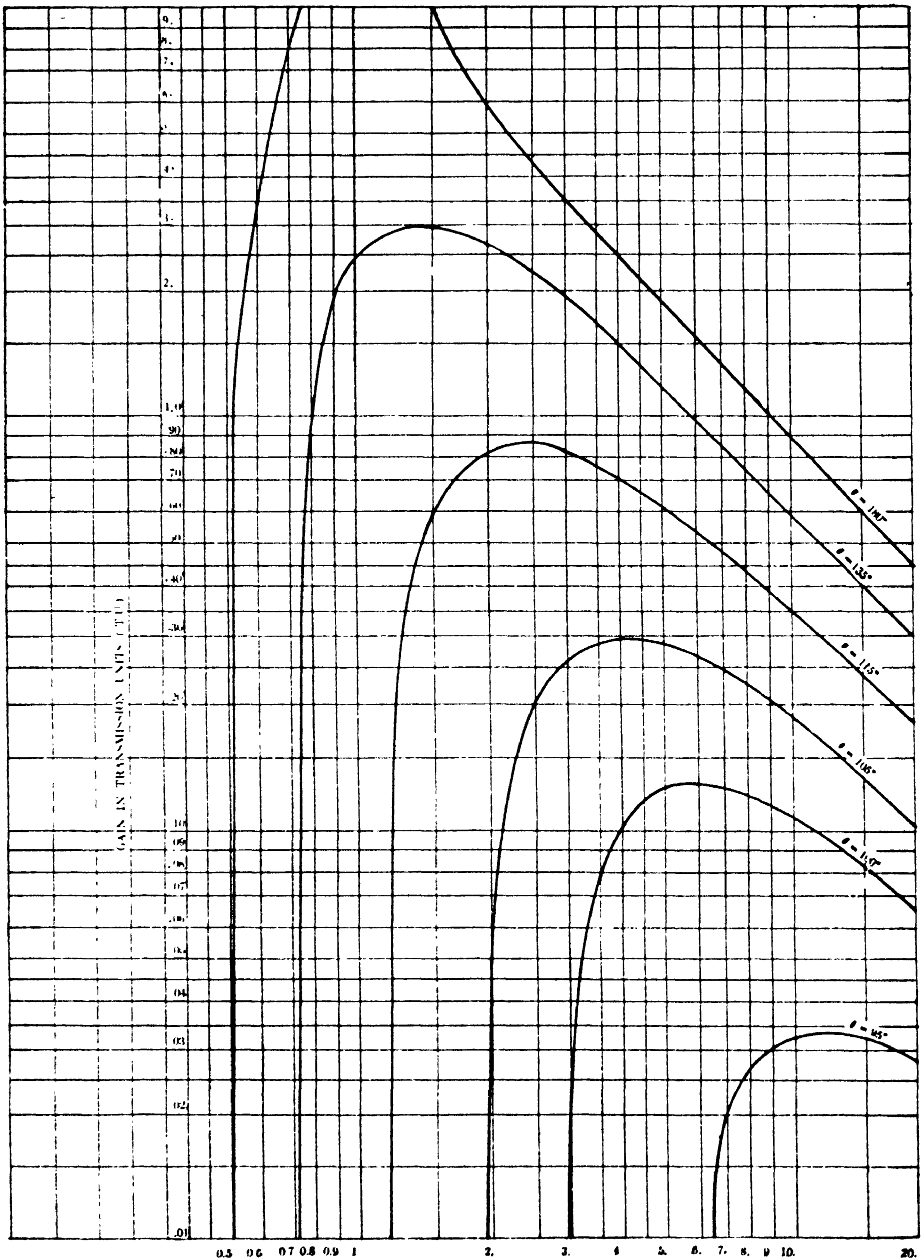


Fig. 5. Curves showing the transmission loss in TU caused by inserting in a circuit a simple series impedance, Z_C, or a simple shunt impedance, Z_S.



$$\left|\frac{Z_S}{Z_1} + \frac{Z_S}{Z_2}\right| \quad \text{or} \quad \left|\frac{Z_1 + Z_2}{Z_C}\right|$$

FIG. 5 (*Continued*)



$$\left|\frac{Z_S}{Z_1}+\frac{Z_S}{Z_2}\right| \quad \text{or} \quad \left|\frac{Z_1+Z_2}{Z_C}\right|$$

FIG. 6. Curves showing the transmission *gain* in TU caused by inserting in a circuit a simple series impedance, Z_C, or a simple shunt impedance, Z_S.

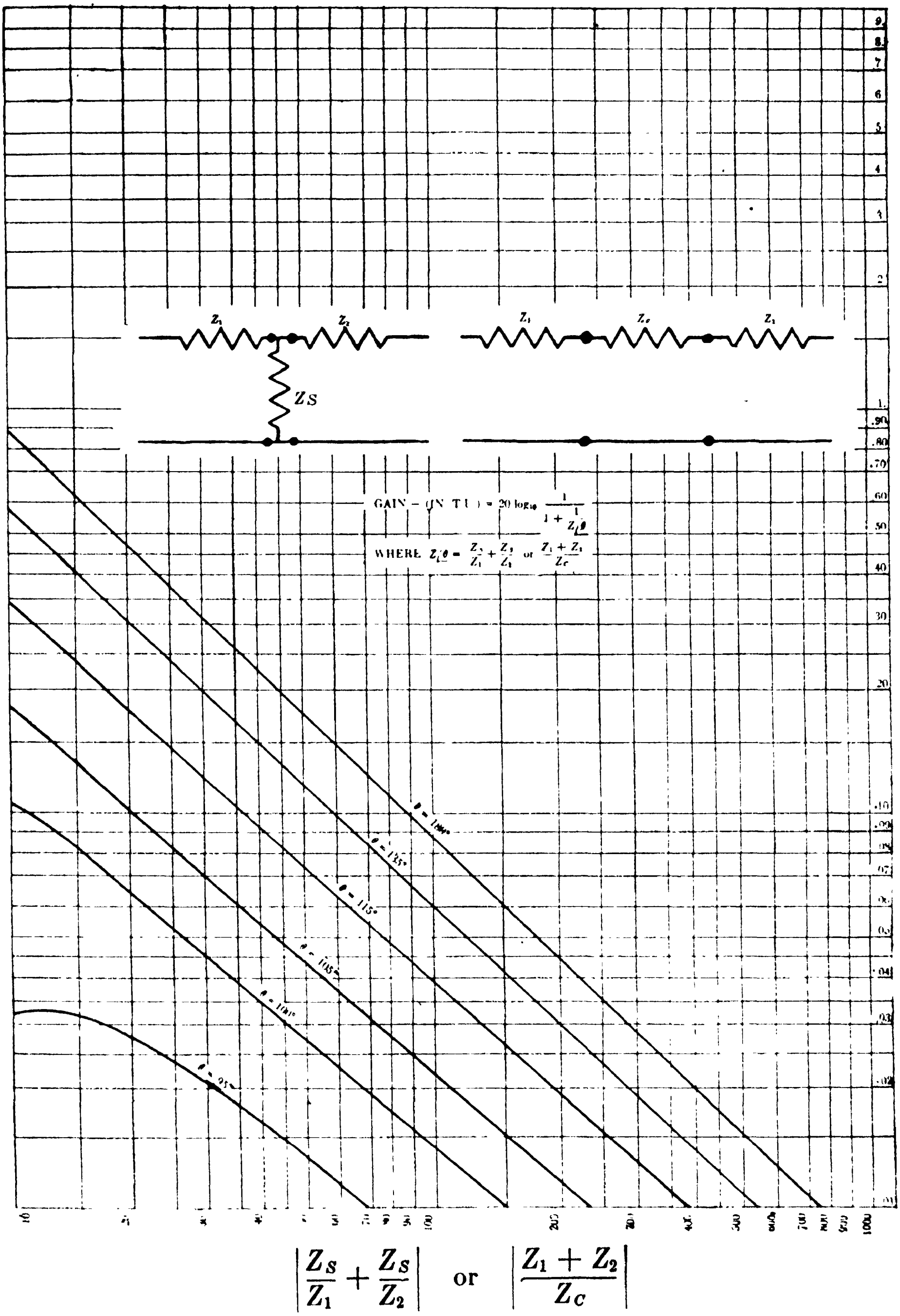


FIG. 6 (*Continued*).

9.4 Graphical Methods.—It will be seen from the foregoing that the process of determining the loss caused by the insertion of any apparatus in a telephone circuit, while not complicated, involves considerable computation and, in view of the fact that such computations have to be made very frequently, certain curves are, in practice, often drawn up to facilitate this work. The more general such curves are made the less labor they save, while on the other hand it is not feasible to draw up curves for each special condition of circuit, frequency, etc.

There are two general types of curves which are in common use, one set permitting the losses or gains due to any series or bridged impedances inserted in any line (and at any frequency) to be determined with the aid of a *small* amount of mathematical work and the other set permitting the determination, without *any* mathematical work, of the losses (at any given frequency) caused by impedances bridged across circuits of known impedances, such as at the middle of or at the end of long uniform lines.

Curves of the first type, as shown in Figs. 5 and 6, are very useful since they make it possible, with only a small amount of computation, to determine the transmission loss or gain caused either by the insertion of series or shunt impedances. Suppose, for example, we wish to bridge a piece of apparatus having an impedance Z_S of 3,640 /90° ohms across the junction of a long No. 12 N.B.S. gauge open wire line having an iterative impedance Z_1 of 715 /12° ohms and a substation circuit whose impedance Z_2 is 640 /30° ohms. We find that the value of $Z_S/Z_1 + Z_S/Z_2$ is approximately 10 /80° and from the curves in Fig. 5 that the corresponding transmission loss is .19 *transmission unit* (TU). Similarly, if an impedance Z_C of 210 /63° ohms is inserted in series between the line and the subscriber's set referred to above, the value of $(Z_1 + Z_2)/Z_C$ is approximately 6 /70° which from the curves given in Fig. 5 corresponds to a transmission loss of .57 TU.

Curves of the second type are shown in Fig. 7 and enable us, as previously stated, to determine at once and *without any calculations* the transmission loss caused by bridging any apparatus (such as relays, retardation coils, etc.) across the specific circuit for which the curves are drawn. For example, if a relay whose impedance is 400 + j1,200 ohms is to be bridged across the end of a long open wire circuit having an impedance in one direction of 711/14° 46′ ohms and in the other direction 728/10° 19′ ohms, we find by reference to the curves in Fig. 7 that the transmission loss caused thereby is 1.48 miles of standard cable. It should be noted that the curves of Fig. 7 are drawn for the *specific values* of the terminal impedances referred to above.

9.5 Calculation of Very Small Transmission Losses.—It is frequently desired to determine the transmission loss which is caused by the insertion either of a small series impedance or a large shunt impedance. Such a determination can properly be made by means of the formulae previously given, but due to errors in slide rule work, etc., it is oftentimes more convenient and more accurate to use formulae of the type which are derived below.

Suppose we have the circuit shown in Fig. 8 and wish to determine the transmission loss caused by the insertion of a small resistance R. It is known that the absolute magnitude of the ratio of the currents after and before inserting the resistance is:

$$\left|\frac{I_1}{I_2}\right| = \epsilon^{-LA_C} = \left|\frac{Z_1 + Z_2}{Z_1 + Z_2 + R}\right| \tag{24}$$

where L is the loss in miles of standard cable and A_C is the attenuation constant of standard cable at the frequency considered. Squaring both sides of the above equation and using the component parts of Z_1 and Z_2,

$$\epsilon^{-2LA_C} = \frac{(R_1 + R_2)^2 + (X_1 + X_2)^2}{(R_1 + R_2 + R)^2 + (X_1 + X_2)^2} \tag{25}$$

Taking the reciprocal of this equation and the logarithm of both sides,

$$2LA_C = \log_\epsilon \frac{(R_1 + R_2 + R)^2 + (X_1 + X_2)^2}{(R_1 + R_2)^2 + (X_1 + X_2)^2} \tag{26}$$

The derivative dL/dR of this equation is

$$\frac{dL}{dR} = \frac{R_1 + R_2 + R}{A_C[(R_1 + R_2 + R)^2 + (X_1 + X_2)^2]} \tag{27}$$

When there is no initial inserted resistance in the circuit, R is zero and equation (27) becomes

$$\frac{dL}{dR} = \frac{R_1 + R_2}{A_C[(R_1 + R_2)^2 + (X_1 + X_2)^2]} = \frac{R_1 + R_2}{A_C|Z_1 + Z_2|^2} \tag{28}$$

The above value of dL/dR is, therefore, the loss in miles of standard cable per ohm of resistance inserted in the circuit.

If a frequency of 796 cycles ($\omega = 5{,}000$) is assumed, the attenuation constant of standard cable, A_C, is .109 and hence equation (28) becomes

$$\frac{dL}{dR} = \frac{R_1 + R_2}{.109|Z_1 + Z_2|^2} \tag{29}$$

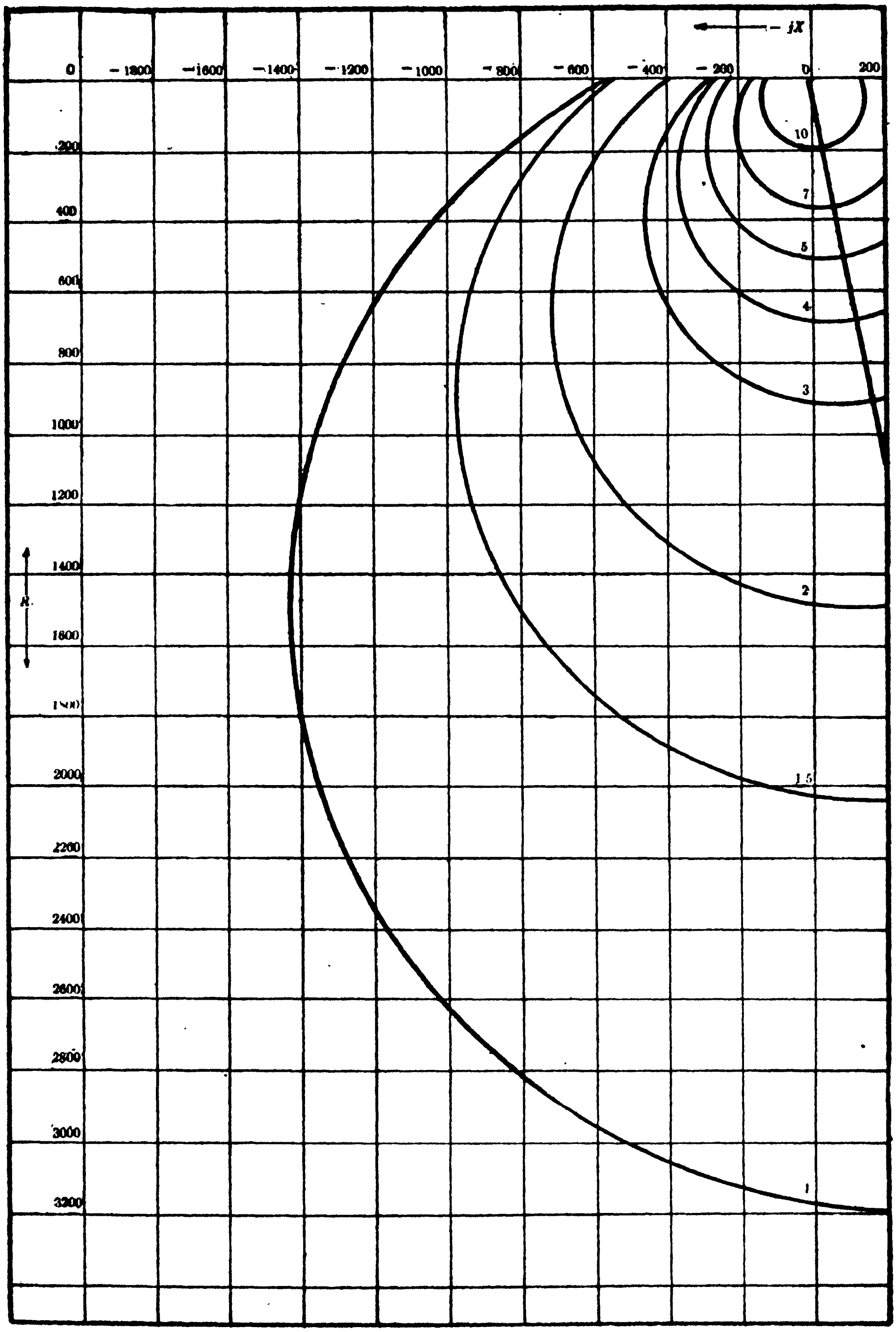
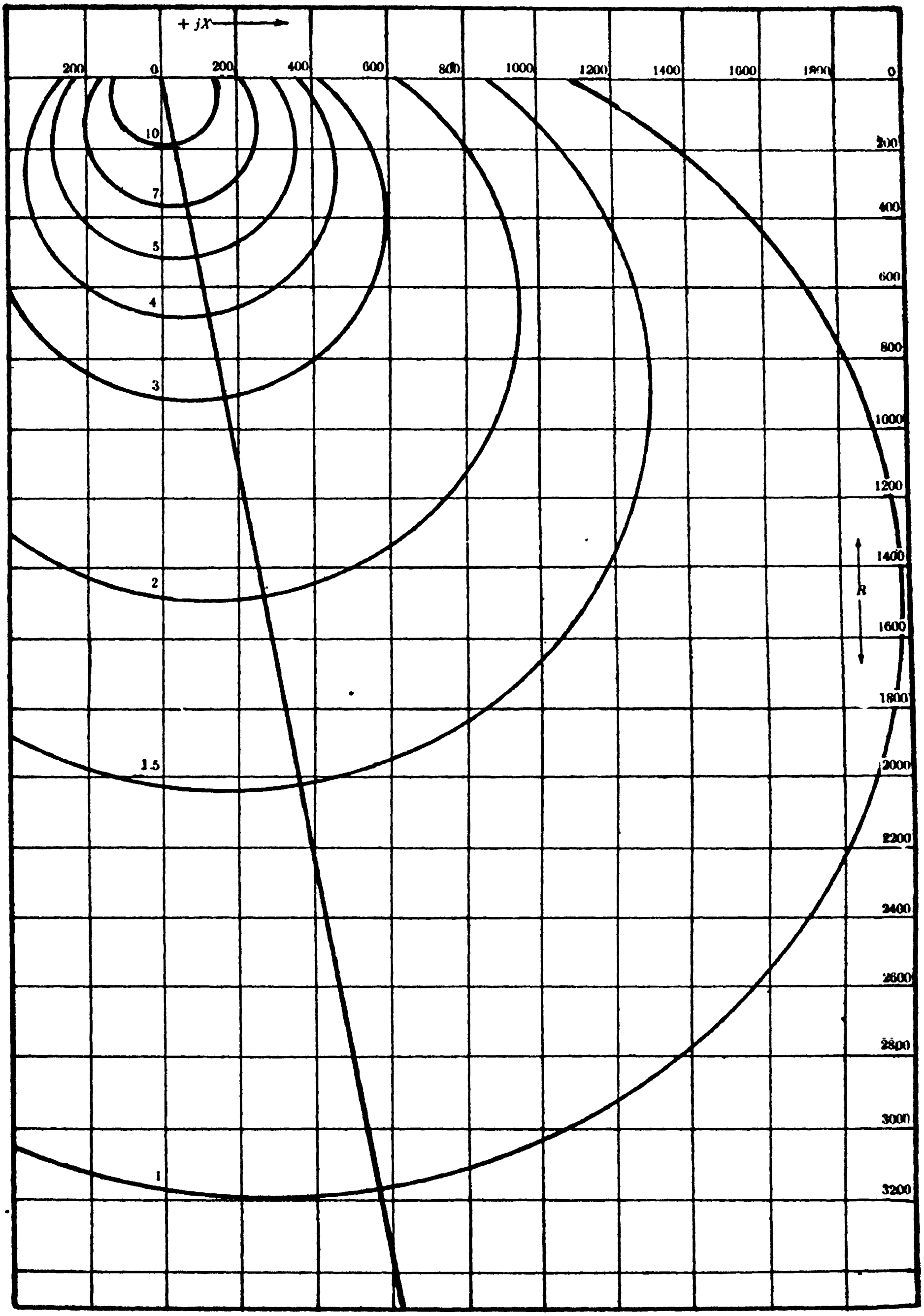


FIG. 7. Curves showing the transmission loss, in miles of standard cable, due to bridging an impedance $(R + jX)$ at the junction of a long non-loaded No. 12 N.B.S.



gauge open wire line and a typical subscriber's loop. Curves assume that the impedance of the line is 711 $\underline{/14°46'}$ ohms and that the impedance of the subscriber's loop is 728 $\underline{/10°19'}$ ohms.

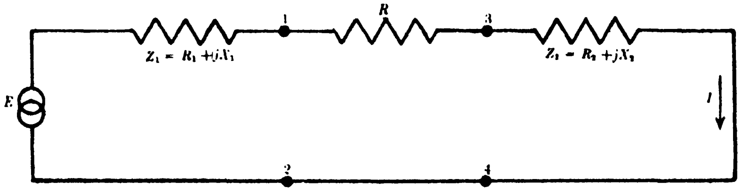


FIG. 8. Circuit in which the transmission loss, due to the insertion of a small resistance *R*, is to be calculated.

As an example of the use of this method, suppose we have the common battery substation circuit shown in Fig. 9 and wish to find the a-c. loss resulting from increasing the effective resistance in the 1–2 winding of the induction coil by a small value—say 10 ohms. Assuming a frequency of 796 cycles, equation (29) gives

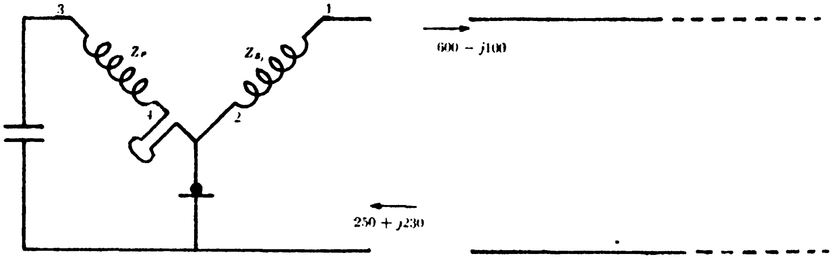


FIG. 9. Common battery substation circuit, illustrating the calculation of the transmission loss due to the insertion of a small series resistance.

$$\frac{dL}{dR} = \frac{250 + 600}{.109[(250 + 600)^2 + (230 - 100)^2]} = .0105 \text{ mile per ohm} \qquad (30)$$

Consequently, for small values of inserted resistance each ohm increase in the 1–2 winding will increase the voice-frequency transmitting (or receiving) loss of the substation set by .0105 mile of standard cable. The loss due to the 10 ohms will, therefore, be .105 mile.

We can, by a similar method, determine the transmission loss caused by shunting a very high impedance across any point of a circuit. For example, consider the circuit shown in Fig. 10. If we let $r/\theta \equiv Z_S/Z_A$

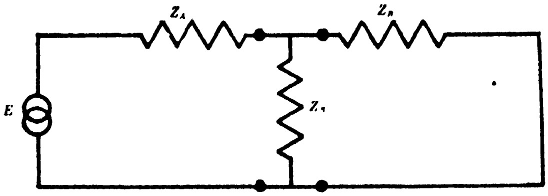


FIG. 10. Circuit illustrating the calculation of the transmission loss due to the bridging of a high impedance Z_S.

$+ Z_S/Z_B$, then it can be shown that the transmission loss, L, in miles of standard cable caused by the bridged impedance Z_S is

$$L = \frac{2.303}{2A_C} \log_{10} \left[1 + \frac{2 \cos \theta}{r} + \frac{1}{r^2} \right] \tag{31}$$

When Z_S is large, and provided θ does not equal ± 90°

$$L \doteqdot \frac{\cos \theta}{A_C r} \tag{32}$$

in which A_C is the attenuation constant of standard cable (.109 at 796 cycles). This last formula holds very closely when r has a value of ten or greater—unless θ is extremely near ± 90°.

CHAPTER X

SUBSTATION CIRCUITS

10.0 If, by some method, a substation circuit can be so arranged that (1) when transmitting, the receiving element is removed from the circuit and (2) when receiving, the transmitting element is removed from the circuit, the transmission efficiency of the circuit may be materially improved. A substation circuit that is capable of such a change is called a *variable substation circuit* in contradistinction to an *invariable substation circuit* in which the circuit is identically the same electrically whether transmitting or receiving.

10.1 Ideal Variable Circuits.—If we are to have a variable substation circuit that is ideal from a transmission point of view, it is evident that, on receiving, all the incoming power which it is possible to absorb from the line, or source of power, must be dissipated in the receiving element and, similarly, on transmitting there must be nothing in the substation circuit but the transmitting element—associated, of course, with such a transformer, etc., as may be found desirable to impart the maximum possible power to the line.

Consider, for example, the simple circuit shown in Fig. 1 in which a transmitting element of impedance Z_T is connected by a transformer to an electrically long line whose impedance, as measured from the terminals of the substation circuit, is Z_L. At the other end of the line is a receiving set consisting simply of a receiving element Z_R associated with a suitable transformer.

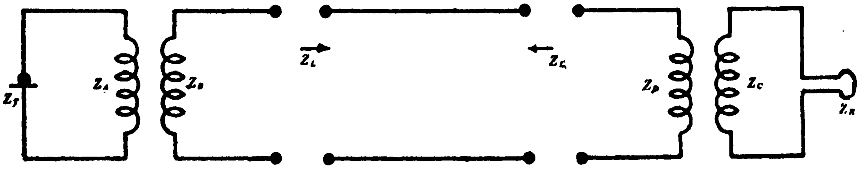


FIG. 1. Simple ideal variable circuit.

10.11 Receiving.—If we now consider the transmitter to be a source of constant a-c. electromotive force, we can, by means of Thévenin's Theorem, reduce the above circuit—in so far as changes in current at the receiving end are concerned—to the circuit shown in Fig. 2.

The problem then, so far as the receiving end is concerned, is to determine what relation must exist between Z_D, Z_C and Z_R in the above circuit in order that there shall be maximum power dissipated in Z_R.

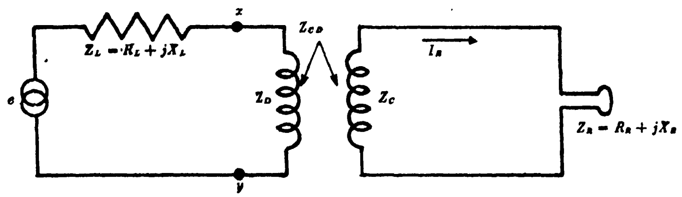


FIG. 2. Equivalent circuit of simple variable substation circuit, when receiving.

Although the answer to this problem has been essentially obtained earlier (see Sect. 7.1), the complete proof will be given again for the sake of clearness.

The current I_R is given by

$$I_R = \frac{eZ_{CD}}{(Z_L + Z_D)(Z_R + Z_C) - (Z_{CD})^2} \tag{1}$$

It will be assumed that the transformer is ideal; i.e., that it has no capacity, resistance or flux leakage and that the self impedance of each winding is infinite. If then we let $r \equiv Z_D/Z_C$ and the mutual impedance $Z_{CD} = \sqrt{Z_C Z_D} = Z_C\sqrt{r}$, equation (1) gives

$$I_R = \frac{e\sqrt{r}}{Z_L + Z_R r} = \frac{e\sqrt{r}}{(R_L + jX_L) + (R_R + jX_R)r} \tag{2}$$

The power dissipated in Z_R will be:

$$|I_R|^2 R_R = \frac{e^2 r R_R}{(R_L + R_R r)^2 + (X_L + X_R r)^2} \tag{3}$$

By the usual process of differentiation, this is found to be a maximum —if r is variable—when

$$r = \frac{Z_D}{Z_C} = \sqrt{\frac{R_L^2 + X_L^2}{R_R^2 + X_R^2}} = \left|\frac{Z_L}{Z_R}\right| \tag{4}$$

Substituting this value of r in (2) gives

$$I_R = \frac{e}{Z_L\sqrt{\left|\frac{Z_R}{Z_L}\right|} + Z_R\sqrt{\left|\frac{Z_L}{Z_R}\right|}} \tag{5}$$

Next insert in series with the line (at the point x, y) a reactance of $-X_L$ ohms. The impedance Z_L will then be effectively R_L ohms.

Similarly, the insertion of a reactance of $-X_R$ ohms in series with the receiving element will make its impedance Z_R become effectively R_R ohms. Under these conditions (5) becomes

$$I_R = \frac{e}{2\sqrt{R_L R_R}} \tag{6}$$

which represents the absolute maximum value of current that can be obtained in a receiver of impedance $Z_R = R_R + jX_R$ ohms when connected to a line having an impedance of $Z_L = R_L + jX_L$ ohms.

Since the impedance of the substation circuit as measured from its terminals (x, y) has been shown to be (see equation (26), Chapter VI),

$$Z = \frac{Z_D}{Z_C} Z_R = rZ_R \tag{7}$$

it is evident from an inspection of (4) and (7) that the impedance of the substation set Z is equal in absolute value to that of the line Z_L provided the set is so designed as to obtain maximum power in the receiving element.

10.12 Transmitting.—In considering the transmitting efficiency of a substation circuit, when connected to an electrically long line, it is only necessary to consider what design of the substation circuit will produce the greatest current at the outgoing end of the line. This is evident since the ratio of the current received at the far end to that starting out on the line will be a constant depending upon the electrical characteristics of the line and the receiving apparatus and is independent of anything that may be done to the circuit of the transmitting substation set. The problem then, in the particular case assumed, is to determine when the substation circuit will produce the maximum current in the line when its terminals are connected to an impedance Z_L and to determine the value of this current.

Referring to Fig. 3 the expression for I_L is

$$I_L = \frac{EZ_{AB}}{(Z_A + Z_T)(Z_B + Z_L) - (Z_{AB})^2} \tag{8}$$

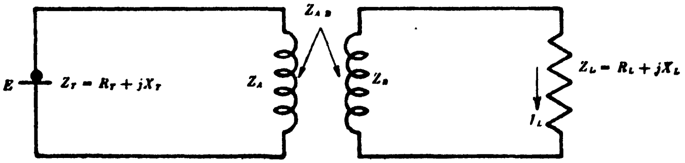


FIG. 3. Circuit of simple variable substation set when transmitting.

From the similarity of this equation with equation (1) it is evident that the current I_L is a maximum when

$$\left|\frac{Z_B}{Z_A}\right| = \left|\frac{Z_L}{Z_T}\right| \tag{9}$$

Moreover, when the reactances of Z_T and Z_L have been annulled, by inserting in series with them reactances of $-X_T$ and $-X_L$ ohms respectively, the value of outgoing current I_L will be

$$I_L = \frac{E}{2\sqrt{R_T R_L}} \tag{10}$$

This is then the maximum value of outgoing current that it is possible to obtain on a line whose impedance is $Z_L = R_L + jX_L$ ohms, if the transmitting element has an electromotive force E and an internal impedance of $Z_T = R_T + jX_T$ ohms.

10.2 Ideal Invariable Circuits—with Unity Power Ratio.—In the case of an invariable substation circuit it is evident that if two-way service is to be had there must be both a transmitting element and a receiving element at each set. Since it is not known as yet what relations must exist between the impedances of the transmitting element, the receiving element, and the line so that the set may have maximum combined transmitting and receiving efficiency, ideal transformers will be assumed to be inserted as shown in the circuit in Fig. 4—in order that we may effectively give to the elements whatever impedances may be found to be the most desirable.

10.21 Transmitting.—Assuming as usual that the transmitter may be considered to be the source of a constant a-c. electromotive force and that the set is to be connected to a long line of impedance Z_L, the circuit is as shown in Fig. 4.

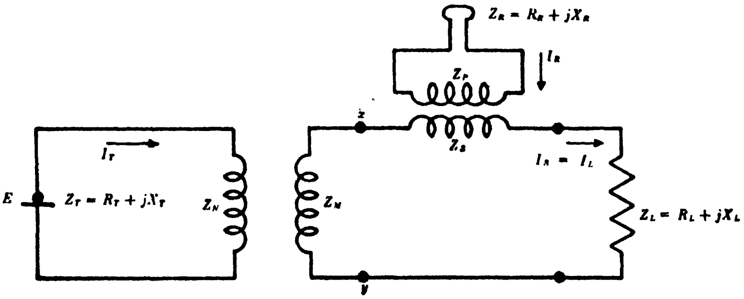


FIG. 4. Schematic of an ideal invariable substation circuit when transmitting.

Assuming ideal transformers, the expression for the current in the line is

$$I_L = \frac{E\sqrt{\dfrac{Z_M}{Z_N}}}{Z_L + \dfrac{Z_M}{Z_N}Z_T + \dfrac{Z_S}{Z_P}Z_R} \tag{11}$$

10.22 Receiving.—On receiving we can assume (by means of Thévenin's Theorem) an electromotive force e acting on the set in series with the impedance Z_L as shown in, Fig. 5.

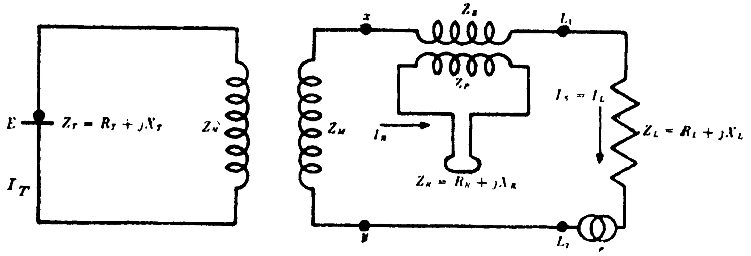


FIG. 5. Schematic of an ideal invariable substation circuit when receiving.

With ideal transformers, the current in the receiver would be

$$I_R = \frac{e\sqrt{\dfrac{Z_S}{Z_P}}}{Z_L + \dfrac{Z_M}{Z_N}Z_T + \dfrac{Z_S}{Z_P}Z_R} \tag{12}$$

10.23 Combined Transmitting and Receiving.—Since the transmitting efficiency depends upon the value of I_L as given by (11) and the receiving depends upon the value of I_R in (12), the combined transmitting and receiving or overall efficiency of the substation circuit will be a maximum when the product of I_R and I_L is a maximum. This product is

$$I_L \times I_R = \frac{Ee\sqrt{\dfrac{Z_M}{Z_N}}\sqrt{\dfrac{Z_S}{Z_P}}}{\left[Z_L + \dfrac{Z_M}{Z_N}Z_T + \dfrac{Z_S}{Z_P}Z_R\right]^2} \tag{13}$$

Assuming the ratio Z_M/Z_N to be variable, and Z_L, Z_R and Z_T to have equal phase angles, this product is a maximum when

$$\frac{Z_M}{Z_N} = \frac{Z_L + \dfrac{Z_S}{Z_P}Z_R}{3Z_T} \tag{14}$$

Similarly, if the ratio Z_S/Z_P is varied, the product of I_R and I_L as given by (13) is a maximum when

$$\frac{Z_S}{Z_P} = \frac{Z_L + \frac{Z_M}{Z_N} Z_T}{3Z_R} \tag{15}$$

Solving (14) and (15) simultaneously,

$$\frac{Z_S}{Z_P} Z_R = \frac{Z_L}{2} \tag{16}$$

and

$$\frac{Z_M}{Z_N} Z_T = \frac{Z_L}{2} \tag{17}$$

Since $(Z_S/Z_P)Z_R$ is the effective impedance of the receiving element and $(Z_M/Z_N)Z_T$ is the effective impedance of the transmitting element, it is evident that the effective impedance of both elements must be equal to each other and that their sum, which is equal to the impedance of the set, must equal the impedance of the line Z_L as measured from the set terminals.

Substituting then the values of (16) and (17) in equations (11), (12) and (13), we get

$$I_L = \frac{E}{2\sqrt{2Z_T Z_L}} \tag{18}$$

$$I_R = \frac{e}{2\sqrt{2Z_R Z_L}} \tag{19}$$

and

$$I_L \times I_R = \frac{Ee}{8Z_L\sqrt{Z_T Z_R}} \tag{20}$$

which are, respectively, a measure of the transmitting, receiving and overall efficiency of a circuit designed for maximum overall (or combined transmitting and receiving) efficiency.

If, as before, the reactances of Z_L, Z_R and Z_T are annulled by inserting in series with these elements reactances of $-X_L$, $-X_R$ and $-X_T$ ohms respectively, the above equations become

$$I_L = \frac{E}{2\sqrt{2R_L R_T}} \tag{21}$$

$$I_R = \frac{e}{2\sqrt{2R_L R_R}} \tag{22}$$

and

$$I_L \times I_R = \frac{Ee}{8R_L \sqrt{R_R R_T}} \tag{23}$$

These last three expressions, representing as they do the maximum values of the transmitted and received currents (as well as of their product) which it is possible to obtain in an ideal invariable substation circuit designed for maximum overall efficiency, furnish a very useful means of determining the relative transmitting, receiving and combined transmitting and receiving efficiency of any type of substation circuit as compared with that of the ideal invariable substation circuit.

We will now consider how the efficiency of the ideal *invariable* circuit compares with that of the ideal *variable* circuit. From (6) and (10) the product of the currents in the ideal variable circuit is

$$I_L \times I_R = \frac{Ee}{4R_L \sqrt{R_R R_T}} \tag{24}$$

A comparison of (23) and (24) shows that in the case of the invariable circuit the product of the currents, representing the total transmitting and receiving efficiency, is only one-half of that obtained when variable sets are used. Since in both cases the impedance of the substation circuit is equal to that of the line, the power received in a system using invariable substation circuits is only one-fourth of that received over a system using variable substation circuits—which is exactly analogous to the result that was obtained in circuits using simply transmitters and receivers (see Sects. 3.1 and 3.2). Also, as previously stated, a loss in current of one-half corresponds to approximately 6.3 miles of standard cable at 800 cycles. Hence, it is evident that *the transmission over any system using invariable substation circuits is inherently* 6 *miles worse than that over a similar system using variable substation circuits.*

10.3 Ideal Invariable Circuits—with Power Ratio *Y*.—If we could increase the transmitting efficiency of a substation circuit by simply causing a corresponding decrease in the receiving efficiency of the circuit, it would be a highly desirable thing to do since the deleterious effects of extraneous line noises would thereby be minimized. In other words, the telephone currents would then be large as compared with extraneous or induced currents. The question then arises as to what effect will any such attempt to boost the transmitting efficiency, at the expense of the receiving efficiency, have upon the overall or combined transmitting and receiving efficiency of the circuit.

10.31 Transmitting and Receiving Efficiency.—As previously shown, the invariable substation circuit has maximum combined transmitting and receiving efficiency when the effective impedance of the transmitting element is equal to that of the receiving element, or, in other words, when the incoming power is equally distributed between the transmitting and receiving elements. Suppose, that when receiving, *Y is the ratio of the power delivered to the transmitter to that delivered to the receiver* and that the problem is to determine how the transmitting, receiving, and combined transmitting and receiving efficiency of a substation circuit varies as Y departs from unity.

Referring again to Fig. 5, the above definition of Y states that

$$Y = \frac{(I_T)^2 R_T}{(I_R)^2 R_R} \tag{25}$$

Assuming as before that the transformers are ideal

$$I_T = I_S\sqrt{\frac{Z_M}{Z_N}} = I_L\sqrt{\frac{Z_M}{Z_N}} \tag{26}$$

and

$$I_R = I_S\sqrt{\frac{Z_S}{Z_P}} = I_L\sqrt{\frac{Z_S}{Z_P}} \tag{27}$$

Substituting the values of (26) and (27) in equation (25)

$$Y = \frac{\dfrac{Z_M}{Z_N} R_T}{\dfrac{Z_S}{Z_P} R_R} \tag{28}$$

Since for maximum efficiency the impedance of the set must equal that of the line and since the reactances of the transmitting and receiving elements must be annulled

$$Z_L = \frac{Z_S}{Z_P} R_R + \frac{Z_M}{Z_N} R_T \tag{29}$$

Solving (28) and (29) simultaneously

$$\frac{Z_M}{Z_N} = \frac{Z_L Y}{R_T(1 + Y)} \tag{30}$$

and

$$\frac{Z_S}{Z_P} = \frac{Z_L}{R_R(1 + Y)} \tag{31}$$

Substituting the value found in (31) in equation (27)

$$I_R = I_S\sqrt{\frac{Z_L}{R_R(1 + Y)}} \tag{32}$$

Since, however, the impedance of the set equals that of the line

$$I_S = \frac{e}{2Z_L} \tag{33}$$

Consequently, (32) becomes

$$I_R = \frac{e}{2\sqrt{Z_L R_R(1 + Y)}} \tag{34}$$

When the reactances of the transmitter and receiver have been annulled, (11) becomes

$$I_L = \frac{E\sqrt{\frac{Z_M}{Z_N}}}{Z_L + \frac{Z_M}{Z_N}R_T + \frac{Z_S}{Z_P}R_R} \tag{35}$$

From (29) and (35)

$$I_L = \frac{E\sqrt{\frac{Z_M}{Z_N}}}{2Z_L} \tag{36}$$

Also from (30) and (36)

$$I_L = \frac{E\sqrt{Y}}{2\sqrt{Z_L R_T(1 + Y)}} \tag{37}$$

If the reactance of the line is annulled by putting in series between it and the set a reactance of $-X_L$ ohms, (34) and (37) become

$$I_R = \frac{e}{2\sqrt{R_L R_R(1 + Y)}} \tag{38}$$

and

$$I_L = \frac{E\sqrt{Y}}{2\sqrt{R_L R_T(1 + Y)}} \tag{39}$$

The product of (38) and (39) gives

$$I_L \times I_R = \frac{Ee\sqrt{Y}}{4R_L(1 + Y)\sqrt{R_T R_R}} \tag{40}$$

When the theoretically most efficient condition holds, i.e., when $Y = 1$, equation (40) reduces, as it should, to equation (23). The last three equations then supply a method for determining the loss or gain in receiving, transmitting or combined transmitting and receiving efficiency due to a variation in the power-distribution ratio Y. For example, the current ratio corresponding to the loss in receiving, due to the fact that the value of Y is not unity, is

$$\frac{I_R}{I_R'} = \sqrt{\frac{2}{1 + Y}} \tag{41}$$

Similarly, the ratio corresponding to the loss in transmitting efficiency is given by

$$\frac{I_L}{I_L'} = \sqrt{\frac{2Y}{1 + Y}} \tag{42}$$

Finally, the ratio of currents corresponding to the combined transmitting and receiving loss is

$$\frac{I_L \times I_R}{I_L' \times I_R'} = \frac{2\sqrt{Y}}{1 + Y} \tag{43}$$

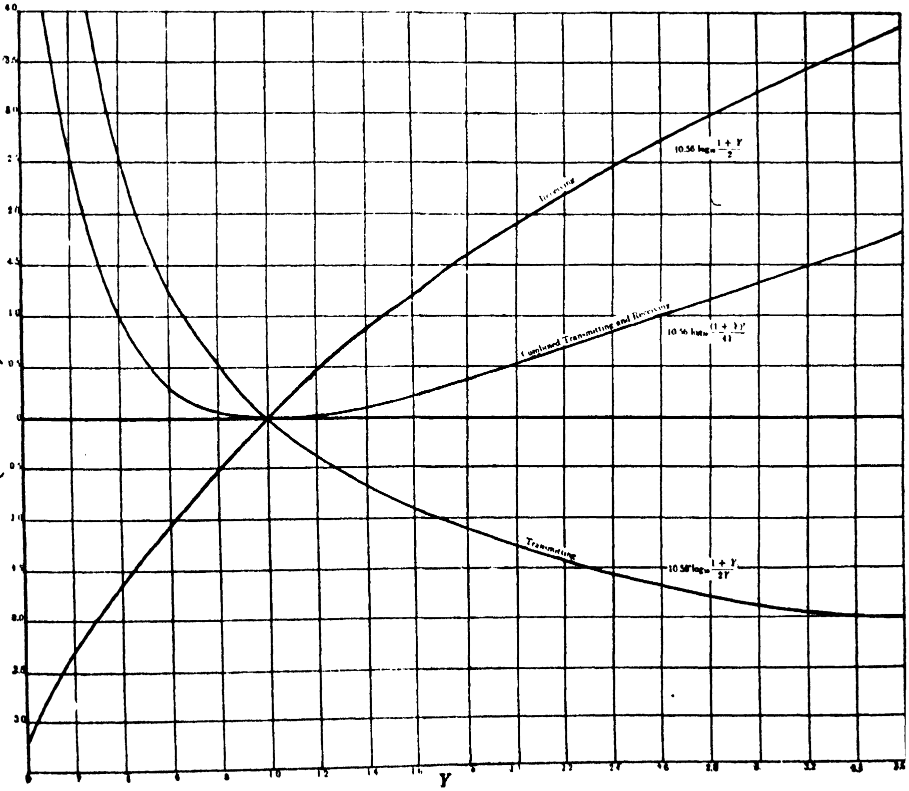


FIG. 6. Substation characteristics as functions of the power-distribution ratio Y.

In order to show more clearly what effect a variation in the power-distribution ratio Y has upon the efficiency of an otherwise ideal substation circuit, there are given in Fig. 6 curves obtained from (41), (42) and (43).

10.32 Side-Tone.—There is one more characteristic of the ideal invariable side-tone circuit which is of interest and that is the amount of current—called side-tone current—which will flow in the receiver of an ideal invariable circuit, whose power-distribution ratio is Y, when an e.m.f. E is acting in the transmitter.

If the values of Z_S/Z_P and of I_L, as given by equations (31) and (39), are substituted in (27) and if the reactance of Z_L is annulled, the current in the receiver, when transmitting—that is, the side-tone current is

$$I_R = \frac{E\sqrt{Y}}{2\sqrt{R_TR_R}(Y+1)} = \frac{E}{4\sqrt{R_TR_R}} \times \sqrt{\frac{4Y}{(Y+1)^2}} \tag{44}$$

If $Y = 1$,

$$I_R = \frac{E}{4\sqrt{R_TR_R}} \tag{45}$$

If then we compute the current in the receiver of any actual subscriber's circuit—when transmitting—and compare it with the side-tone current flowing in the ideal invariable subscriber's circuit—as given in (45)—we have a convenient and definite standard of reference by which the side-tone efficiency of any actual subscriber's circuit can be expressed. There is then a definite standard consisting of an ideal invariable circuit with which the transmitting, receiving, or side-tone efficiency of any actual substation circuit can be compared.

10.4 Efficiencies of an Actual Substation Circuit in Terms of the Ideal Invariable Circuit.—To illustrate the use of the formulae of the preceding sections and the method outlined above for determining the transmitting, receiving, overall, and side-tone efficiencies of any actual substation circuit as compared with those of the ideal invariable circuit, consider the standard common battery circuit, the formulae for which, derived in the usual way by Kirchhoff's Laws, are given in Appendix E. If it is assumed that the common battery set uses an induction coil receiver, transmitter, and a condenser and is directly connected to a long No. 12 N.B.S.G. open wire line, the circuit constants at 796 cycles ($\omega = 5{,}000$) might be approximately as follows:

$$Z_L = 687 - j181 \qquad Z_M = 15.8 + j494$$
$$Z_S = 33.8 + j598 \qquad Z_C = 0 - j100$$
$$Z_P = 40.3 + j424 \qquad Z_R = 140 + j160$$
$$Z_T = 40 + j0$$

TABLE I

When	Quantity	Formulae used	Computed value
Transmitting	I_L, C.B. circuit	(53) of Appendix E	$\lvert I_L \rvert = .001892E$
	I_L, ideal circuit	Equation (21)	$\lvert I_L \rvert = .002133E$
	Ratio of line currents		$r = .889$
	Efficiency of C.B. circuit, in miles below ideal circuit	$L = 21.13 \log_{10}\frac{1}{r}$	$L = 1.1$ miles
Transmitting	I_R, C.B. circuit	(54) of Appendix E	$\lvert I_R \rvert = .00222E$
	I_R, ideal side-tone circuit	Equation (45)	$\lvert I_R \rvert = .00334E$
	Ratio of side-tone currents		$r = .664$
	Volume of side-tone, in C.B. circuit, below that in an ideal side-tone circuit	$L = 21.13 \log_{10}\frac{1}{r}$	$L = 3.8$ miles
Receiving	I_R, C.B. circuit	(57) of Appendix E	$\lvert I_R \rvert = .000938e$
	I_R, ideal circuit	Equation (22)	$\lvert I_R \rvert = .00114e$
	Ratio of received currents		$r = .823$
	Efficiency of C.B. circuit, in miles, below ideal circuit	$L = 21.13 \log_{10}\frac{1}{r}$	$L = 1.8$ miles

Using the formulae for the standard common battery circuit as shown in Appendix E and comparing the resulting currents with those which would be obtained in the ideal invariable side-tone circuit, the data shown in Table I are obtained.

From the table it is evident that, at 796 cycles ($\omega = 5{,}000$) and with the circuit constants assumed, the efficiency of the standard common battery substation circuit is, on transmitting, 1.1 miles and, on receiving, 1.8 miles or in combined transmitting and receiving, or overall efficiency, is 2.9 miles below that of the ideal invariable substation circuit. In other words, if an infinite amount of money were to be spent on the circuit or upon the induction coil or condenser, etc., we could improve the overall efficiency of the above circuit by only 2.9 miles of standard cable. *Consequently, if any very large gain (such as 3 or more miles) is ever to be made in the overall efficiency of the present type of standard common battery set, it must be accomplished by the use of instruments that are inherently more efficient and not by the use of any new or improved type of invariable substation circuit.*

10.5 Side-Tone Circuits.—It is evident, from the preceding, that the side-tone of the standard common battery circuit, having the constants assumed, is 3.8 miles below that of the ideal invariable side-tone circuit. A little consideration will show that this is a natural characteristic which is to be expected of a *series type* of substation circuit in which the impedance of the set is *lower* than that of the line to which it is connected. *By a series type of side-tone substation circuit is meant one in which the various elements (transmitter, receiver and line) are all effectively in series with each other as contrasted with a parallel type of circuit in which the elements are all effectively in parallel with each other.* A list of all series and parallel types of side-tone substation circuits using not more than one transformer is given in Fig. 7. It may be noted that thirteen of the circuits are of the series type while forty-one circuits are of the parallel type.

The classification of most of these circuits into the series and parallel types is obvious, such as, for example, the standard local battery form of substation circuit which is given in Fig. 7 as circuit No. 2 of Type A. On the other hand, there are some circuits such as those shown as circuits No. 5, No. 6, and No. 7 of Type A where the series relation is not quite so obvious. If, however, we inspect the formulae for the impedance of any of these circuits their proper type is clearly shown. For example: circuit No. 6 of Type A, which is the standard common battery type of substation circuit, can readily be seen to be of the series type by referring to formula (59) for this circuit, as given in Appendix E. That the series classification is correct for this circuit is proved by the fact that the impedance of the circuit is equal to an impedance ratio Z_S/Z_P multiplied by the receiver impedance, Z_R, *plus* another impedance ratio multiplied by the transmitter impedance, Z_T.

It is evident that if the line impedance is large as compared with the impedance of the substation circuit, the side-tone in the series type of circuit will be relatively small while in the parallel type of circuit it will be relatively large. Similarly, the opposite will hold true if the line impedance is small as compared to the impedance of the substation circuit.

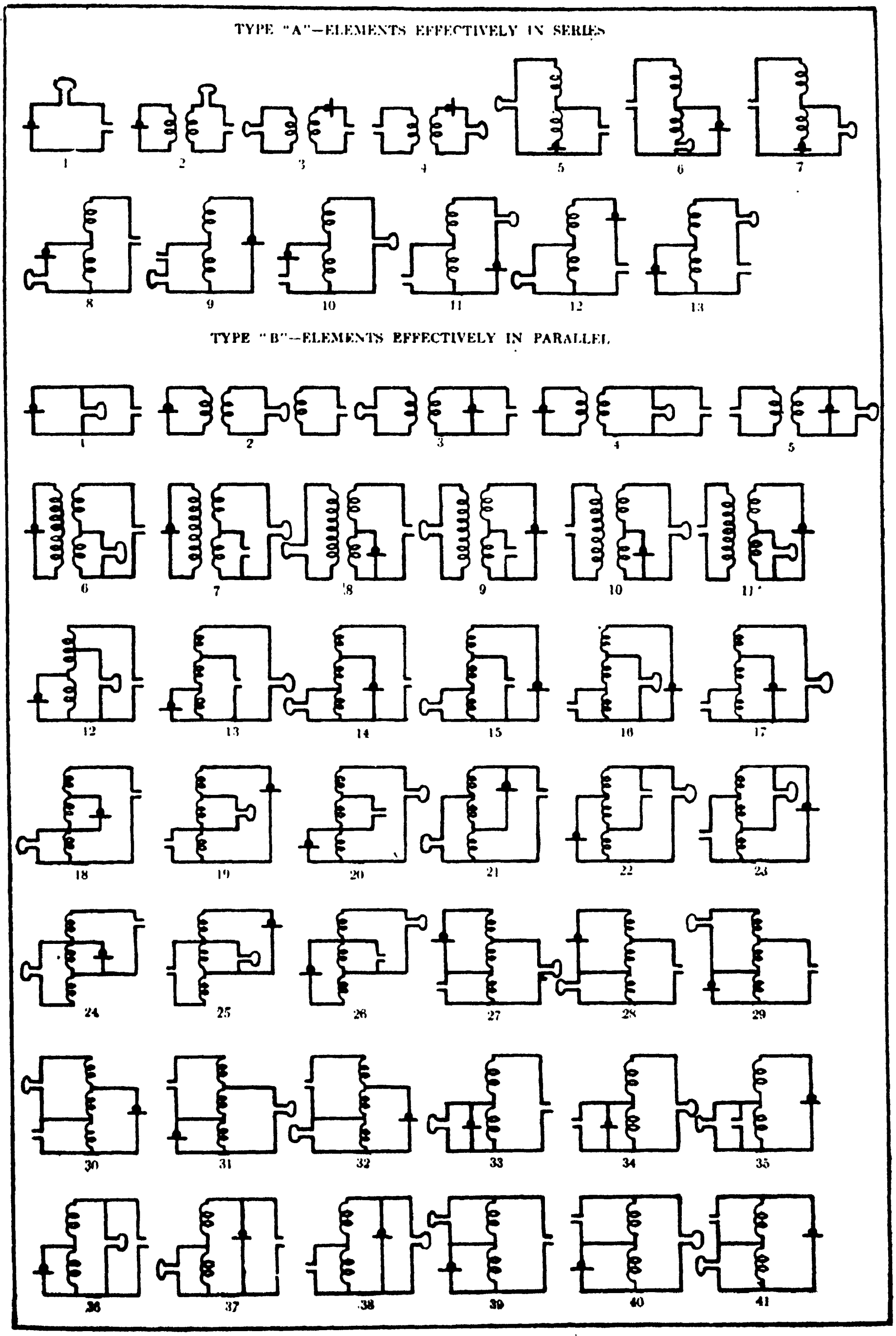


FIG. 7. List of ideal invariable side-tone substation circuits.

The relative advantages of the series and parallel types of side-tone circuits may be easily understood by a little consideration. In both types of circuits—if they are ideal and have a unity power ratio, Y—the power dissipated in the receiver, when transmitting, is one-half of that dissipated in the line. Possibly the most interesting characteristics which distinguish the two types of circuit are (1) that in the series type of circuit, the impedance from the transmitter (or receiver) terminals is (in the ideal case where the power ratio $Y = 1$) three times that of the transmitter (or receiver) impedance, while in the ideal parallel type of circuit the impedance is one-third that of the transmitter (or receiver) impedance; and (2) that while the series and parallel types of circuits both have theoretically the same high frequency efficiency—namely, that of the ideal invariable substation circuit—the series type of circuit is, in most practical cases, somewhat more efficient. This is due to the fact that most carbon button transmitters have a lower effective resistance in their quiet condition than in their talking condition. The result is that they produce what is effectively a variable substation circuit—i.e., the a-c. constants of the circuit are not the same in the transmitting condition as they are in the receiving condition. This variable effect obviously tends to aid the receiving efficiency of a series type of circuit and to cut down the receiving efficiency of the parallel type of circuit. The quantitative effect of this change is quite appreciable and usually gives the series type of circuits a material practical advantage over the parallel type of side-tone substation circuits.

10.6 Anti-Side-Tone Circuits.—*An anti-side-tone substation circuit is one in which there is, in the ideal case, no power dissipated in the receiver when an e.m.f. is generated in the transmitter.* Anti-side-tone circuits are not, as is sometimes supposed, a new development in the telephone art, but have long been in use commercially. Probably the most widely known anti-side-tone circuit is that which is in wide use for common battery operators' sets. This circuit has been used commercially for over twenty years and in many ways is a very satisfactory form of anti-side-tone circuit.

The number of possible forms of anti-side-tone substation circuits is almost innumerable,* depending upon what assumptions are made regarding the number of terminals to the various elements, the number of induction coils or transformers, etc. All efficient invariable anti-side-tone circuits using 2-terminal elements require four of such elements (line, receiver, transmitter and balancing network) or one more element than is required in an equally efficient invariable side-tone circuit. The necessity for this extra element—which is the *balancing network*—may be perhaps most clearly seen by considering some of the best-known forms of anti-side-tone circuits as developments of the well-known Wheatstone bridge circuit.

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* G. A. Campbell, who invented the anti-side-tone substation circuit, using a single induction coil, has shown that with certain assumptions there are over one-half million of such circuits. See article entitled "Maximum Output Networks for Telephone Substation and Repeater Circuits," by G. A. Campbell and R. M. Foster, *Transactions of the A. I. E. E.*, Vol. XXXIX (1920).

Referring to Fig. 8A, there is represented an ordinary Wheatstone bridge. If there is to be no current in the galvanometer Z_G, the potential at the junction of the arms Z_C and Z_D must be the same as that at the junction of the arms Z_A and Z_B. This condition is satisfied when the current (I_1) in arm Z_C is equal to that in arm Z_D and when the current in arm Z_A (I_2) is equal to that in arm Z_B. Then

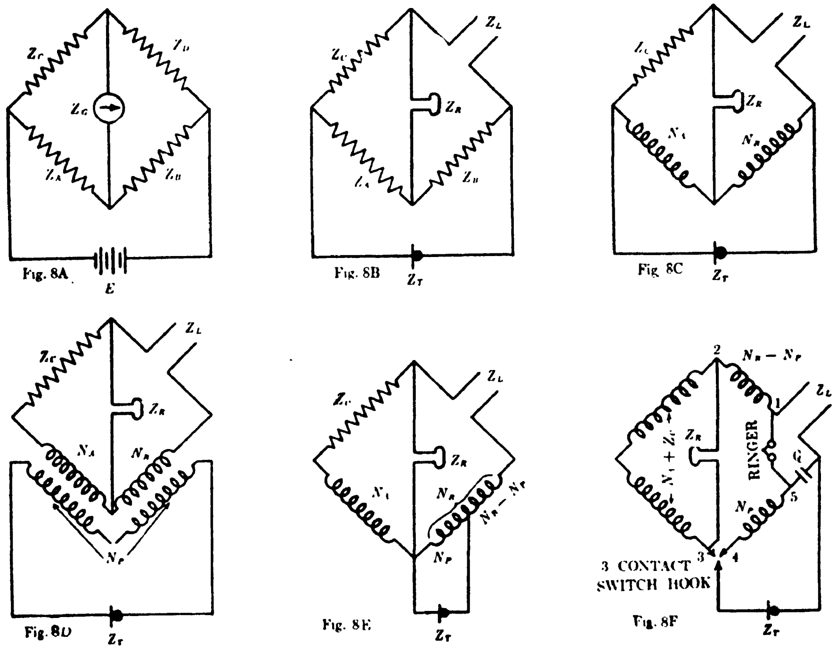

FIG. 8. Development of typical anti-side-tone substation circuits from the Wheatstone bridge circuit.

$$I_1 Z_C = I_2 Z_A; \qquad I_1 Z_D = I_2 Z_B \tag{46}$$

or

$$\frac{I_1}{I_2} = \frac{Z_A}{Z_C} = \frac{Z_B}{Z_D} \quad \text{or} \quad \frac{Z_A}{Z_B} = \frac{Z_C}{Z_D}$$

which is the relation that must exist if the galvanometer current is to be zero. This is the relation given in equation (26) of Chapter V.

Passing to Fig. 8B, suppose the impedance of a line Z_L is substituted for the arm Z_D, a receiver of impedance Z_R for the galvanometer Z_C and a transmitter of impedance Z_T for the source of e.m.f. From the relations developed from the simple Wheatstone bridge circuit it is evident that when an e.m.f. is generated by the transmitter there will be no current in the receiver if the impedances are such that $Z_A/Z_B = Z_C/Z_L$. There will, however, evidently be current in the line. It is also evident that with an e.m.f. in the line, current will pass through the receiver—so that incoming conversation can be heard. Such an arrangement, therefore, may be used as a telephone substation circuit and it will have the very desirable feature of eliminating side-tone provided the conditions of balance noted above can be satisfied at all telephonic frequencies. Actually, of course, it is impossible to secure an exact balance over the entire range of voice frequencies. For any given line, however, the balance can be so adjusted that the side-tone is quite small.

When transmitting with a circuit of this type, however, energy is evidently wasted in the arms Z_A and Z_B as well as in Z_L and Z_C. This can be avoided if we substitute the windings, N_A and N_B, of a transformer or induction coil as is shown in Fig. 8C. The required relation for no side-tone will still be satisfied if the impedances are such that $N_A/N_B = Z_C/Z_L$. If N_A and N_B were windings on an ideal transformer (that is, if they had no effective resistance), there would be no energy wasted in N_A and N_B—the whole of it being consumed in Z_C and Z_L. In an actual transformer or induction coil, there will be only a slight loss due to the resistance of these windings.

Using a separate winding for the transmitter, as shown in Fig. 8D, evidently does not change the " bridge " action of the circuit shown in Fig. 8C, but does allow the impedance of any transmitter to be effectively *stepped up or down* in any desired ratio by a proper choice of N_P. This, of course, is a very desirable feature as it enables a transmitter of any resistance to be used efficiently. The circuit in the form here shown is one of the most desirable types of anti-side-tone substation circuits for local battery use. In an actual set employing this circuit, the element Z_C may consist of a non-inductive resistance, in which case it may be combined with the resistance of the winding N_A.

Instead of the two-winding coil used in the preceding circuit, an auto-transformer type of coil may be used, as shown in Fig. 8E. The transmitter is connected across the same number of turns N_P as in the preceding circuit and hence the action is essentially the same. The use of such an auto-transformer arrangement will obviously give a slightly more efficient circuit than that shown in Fig. 8D.

Fig. 8F shows the circuit of Fig. 8E adapted for common battery use. It is one of the most desirable forms of common battery anti-side-tone circuits. This adaptation differs from the circuit shown in Fig. 8E in minor details only, the differences being as noted below:

(1) Arm Z_C, in the circuit as developed, being a pure resistance, is combined with winding N_A giving one winding of high resistance.

(2) A condenser Q is added to force all the direct current from the line to flow through the transmitter.

(3) Winding $(N_B - N_P)$ has been interchanged with the line (with which it is in series) in order to reduce the number of terminals required on the induction coil from 6 to 5.

(4) A ringer has been added.

(5) Switch hook contacts are shown.

The analysis given above for two of the preferred forms of local and common battery anti-side-tone circuits is, in general, applicable to any other form of anti-side-tone substation circuit. Some anti-side-tone circuits, however, are extremely complicated and their proper design and efficiency characteristics can only be determined by a rigid mathematical analysis. The specific method generally employed in designing and determining the efficiency characteristics of such circuits is first to set up, by Kirchhoff's Laws, the formulae for the currents in the various elements on the assumption (1) of an e.m.f. E in the transmitter, and (2) an e.m.f. e in the line. The conditions for no current in the receiver —with an e.m.f. in the transmitter—and for no current in the balancing network, when there is an e.m.f. in the line, are next determined. The second requirement mentioned above, namely, that there shall be no current in the balancing network when receiving, has been shown by Campbell to be a necessary requirement in order that the overall efficiency of the anti-side-tone circuit may be equal to that of the ideal invariable circuit. The impedance of the set is then calculated from the equations and is equated to the line impedance, Z_L, as measured from the terminals of the set. From these three conditions and the given value of the power ratio Y it is possible to determine the ratios of the induction coil windings and the values of the impedances of the transmitter, receiver, and balancing network which must obtain if the circuit is to have the efficiency of the ideal invariable substation circuit. Although an invariable anti-side-tone circuit can theoretically be so designed as to have the same efficiency as an invariable side-tone circuit, it will actually, under most commercial conditions, be slightly lower due to the fact that the *two conjugacy conditions—i.e. of no current in the receiver when transmitting and no current in the balancing network when receiving*—cannot rigorously be met for all of the different line conditions, frequencies, and similar factors encountered in commercial practice.

That the ideal invariable anti-side-tone circuit has the same overall efficiency as that of the invariable side-tone circuit may be seen when it is remembered that in both circuits one-half of the total incoming power is dissipated in the receiver and one-half is wasted in the transmitter. On the other hand, when transmitting, it can be proved that in either the ideal parallel or series type of side-tone substation circuit only two-thirds of the power given out by the transmitter is absorbed by the line, the remaining one-third of the power being absorbed by the receiver. It will, however, be remembered that the transmitter in the ideal invariable side-tone circuit is working into an impedance which is either one-third of its own impedance or three times its own impedance —depending upon whether or not the circuit is of the parallel or series type. With such a ratio (3 : 1) it can be seen that the transmitter in an ideal invariable side-tone circuit is only giving out (see formula (13) of Chapter VII)

$$\frac{4r}{(1 + r)^2} = \frac{4 \times 3}{(1 + 3)^2} = \frac{3}{4} \tag{48}$$

of the maximum possible power which the transmitter is capable of delivering to an external load. Hence, in the side-tone circuits, the actual power delivered to the line when transmitting is 2/3 × 3/4 = 1/2 of the total power which the transmitter is capable of delivering to an external load. This is, however, exactly identical to the power delivered to the line in an anti-side-tone circuit—in which the resistance looking away from the transmitter (or receiver) terminals is equal to that of the transmitter (or receiver) so that the power absorbed from the transmitter is the maximum power which the transmitter can deliver to an external load. Of this maximum power, however, one-half is dissipated in the line and the other half in the balancing network. When transmitting, therefore, the power delivered to the line is the same in either case—the only difference being that an anti-side-tone circuit draws the maximum possible power which it is possible to absorb from the transmitter while the side-tone circuit draws only three-fourths of this maximum power.

CHAPTER XI

TREATMENT OF PASSIVE NETWORKS AS EQUIVALENT SMOOTH LINES

11.0 The transmission properties of passive networks may often be best determined by considering them as equivalent to lines having smoothly distributed constants. These networks will be divided into two general classes depending upon the symmetry of their structures.

11.1 Properties of Recurrent Symmetrical Structures.—Many structures, such as circuits having smoothly distributed constants or circuits loaded with series or shunt loading coils, may probably best be studied by considering them as special cases of a generalized recurrent structure such as is shown in Fig. 1. It is evident that such a structure can be looked upon as being composed of an infinite number of identical *T* or Π networks.

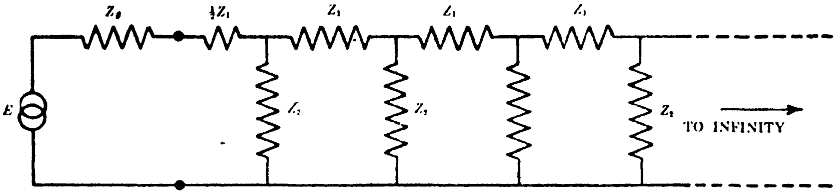

FIG. 1. Recurrent structure, of infinite length, composed of series and shunt impedances.

11.11 Iterative Impedance and Propagation Constant.—Consider, therefore, the characteristics of an infinite length of such a structure, when regarded as made up of an infinite number of symmetrical *T* networks (see Fig. 2). The impedance, Z_K, at the beginning of such an infinite structure is obviously the same as it would be if the first section were omitted—since the structure would still be infinite in length.

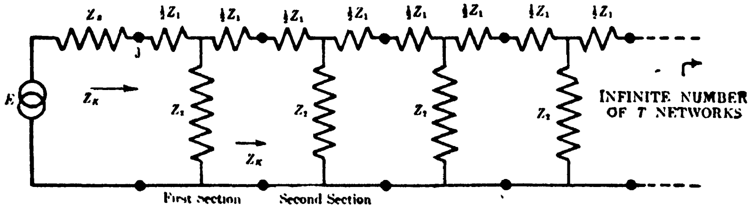

FIG. 2. Recurrent structure of infinite length regarded as made up of symmetrical *T* networks.

Consequently, the value of the impedance, Z_K, at the beginning of the first section must be equal (that is, closer than any assignable amount) to the impedance Z_K in which it is terminated.

The first section, then, can evidently be represented (see Fig. 3) as

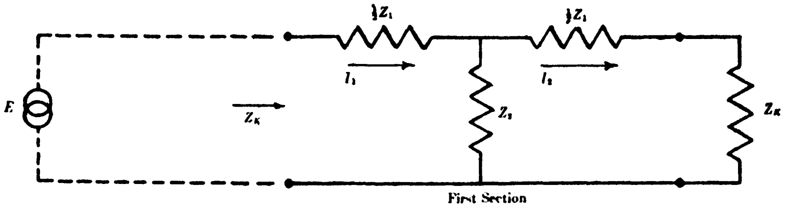

FIG. 3. Single T network terminated in its iterative impedance Z_K.

having an input impedance Z_K and being terminated in the same impedance, viz., Z_K.

From this figure we evidently have

$$Z_K = \frac{Z_1}{2} + \frac{Z_2\left[\frac{Z_1}{2} + Z_K\right]}{Z_2 + \frac{Z_1}{2} + Z_K}$$

or

$$Z_K = \sqrt{Z_1Z_2\left[1 + \frac{1}{4}\frac{Z_1}{Z_2}\right]} \tag{1}$$

Since the *iterative impedance of a structure is defined as the vector ratio of the applied e.m.f. to the resultant steady state current upon a line of infinite length and uniform structure or upon a network composed of an infinite number of recurrent structures,* the above value of Z_K is evidently the *iterative impedance* of the structure. The particular value of the iterative impedance obtained in equation (1) is called the *mid-series iterative impedance*—inasmuch as it is the impedance that obtains at the middle of the series arm Z_1.

In the first section the ratio of the current entering the section, I_1, to that leaving the section, I_2 is

$$\frac{I_1}{I} = \frac{\frac{Z_1}{2} + Z_2 + Z_K}{Z_2} \tag{2}$$

Putting the value of Z_K, as found in (1), in the above equation

$$\frac{I_1}{I_2} = \frac{\dfrac{Z_1}{2} + Z_2 + \sqrt{Z_1 Z_2 + \dfrac{Z_1^2}{4}}}{Z_2} \tag{3}$$

Since the first section is typical of all the other sections (such as the second section, etc.), it is evident that the above current ratio is the same for each section. The natural logarithm of this ratio is called the *propagation constant* P of the structure, that is,

$$P \equiv \log_\epsilon \frac{I_1}{I_2} = \log_\epsilon \frac{\dfrac{Z_1}{2} + Z_2 + \sqrt{Z_1 Z_2 + \dfrac{Z_1^2}{4}}}{Z_2} \tag{4}$$

In other words, *the propagation constant per unit length of a uniform line or per section of a line of recurrent structure is defined as the natural logarithm of the vector ratio of the steady state currents at two points separated by a unit length in a uniform line of infinite length or at two successive corresponding points in a line of recurrent structure of infinite length. The ratio is determined by dividing the value of the current at the point nearer the transmitting end by the value of the current at the point more remote.* In a *symmetrical* structure, the real part of the propagation constant is called the *attenuation constant* ($\equiv A$) while the imaginary part is called the *phase constant* or the *wave length constant* ($\equiv B$).

If $U \equiv 1 + (Z_1/2Z_2)$, (4) becomes

$$P = \log_\epsilon (U + \sqrt{U^2 - 1}) \tag{5}$$

But from the definitions of the hyperbolic functions, viz.,

$$\sinh x \equiv \frac{1}{2}(\epsilon^x - \epsilon^{-x}), \qquad \cosh x \equiv \frac{1}{2}(\epsilon^x + \epsilon^{-x}), \qquad \tanh x \equiv \frac{\sinh x}{\cosh x}$$

it can be easily shown [*] that

$$\begin{aligned} P = \log_\epsilon (U + \sqrt{U^2 - 1}) &= \cosh^{-1} U = \sinh^{-1} \sqrt{U^2 - 1} \\ &= \tanh^{-1} \sqrt{\frac{U^2 - 1}{U^2}} = 2 \sinh^{-1} \sqrt{\frac{U - 1}{2}} \\ &= 2 \cosh^{-1} \sqrt{\frac{U + 1}{2}} = 2 \tanh^{-1} \sqrt{\frac{U - 1}{U + 1}} \end{aligned} \tag{6}$$

Whence, putting the assumed value of U in the various expressions in

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[*] For proofs see Appendix C.

(6), the following important relations are obtained

$$\begin{aligned} P &= \log_e \frac{\frac{Z_1}{2} + Z_2 + \sqrt{Z_1 Z_2\left(1 + \frac{1}{4}\frac{Z_1}{Z_2}\right)}}{Z_2} = \cosh^{-1}\left(1 + \frac{Z_1}{2Z_2}\right) \\ &= \sinh^{-1} \frac{\sqrt{Z_1 Z_2\left(1 + \frac{1}{4}\frac{Z_1}{Z_2}\right)}}{Z_2} = \tanh^{-1} \frac{\sqrt{Z_1 Z_2\left(1 + \frac{1}{4}\frac{Z_1}{Z_2}\right)}}{\frac{Z_1}{2} + Z_2} \qquad (7) \\ &= 2 \sinh^{-1} \frac{1}{2}\sqrt{\frac{Z_1}{Z_2}} = 2 \cosh^{-1} \sqrt{1 + \frac{1}{4}\frac{Z_1}{Z_2}} = 2 \tanh^{-1} \frac{1}{\sqrt{1 + 4\frac{Z_2}{Z_1}}} \end{aligned}$$

Since the ratio I_1/I_2 is the same for every section (see remarks following equation (3)), it is evident that if 2 (or N) sections are taken as the unit periodic or recurrent structure, the ratio of the current entering to the current leaving such a unit will be $(I_1/I_2)(I_1/I_2) = (I_1/I_2)^2$ or in the general case $(I_1/I_2)^N$. But $\log_e (I_1/I_2)^N = N \log_e (I_1/I_2)$. *Hence, the total propagation constant of N identical sections is N times the propagation constant of 1 section—or the total propagation constant of N sections ($\equiv P_N$) is NP, where P is the propagation constant per section.*

It is also important to note, from any of the last three expressions of equation (7), that *the propagation constant of any structure of the ladder or series-shunt type (the names given to the generalized type of structure shown in Fig. 1) depends only upon the vector ratio of Z_1/Z_2.* Consequently, it is possible (see Fig. 11 of Appendix E) to plot curves which will give the value of the attenuation constant A per section of any such structure whose vector ratio Z_1/Z_2 is known. Similar curves (see Fig. 12 of Appendix E) make it possible to find the phase constant B per section of any structure whose vector ratio Z_1/Z_2 is known. Such curves are very convenient and make it possible to determine, with only a slight amount of calculation, the complete propagation characteristics of any uniform or periodic structure, such as a line having smoothly distributed constants, a loaded line, wave filter, etc., without referring to tables of complex hyperbolic functions.

If the structure shown in Fig. 1 is considered to be made up of Π sections (see Fig. 4), each section may be regarded as terminating in a *mid-shunt iterative impedance* ($\equiv Z_K'$). The value of this mid-shunt iterative impedance may evidently be derived from a consideration of the circuit shown in Fig. 5.

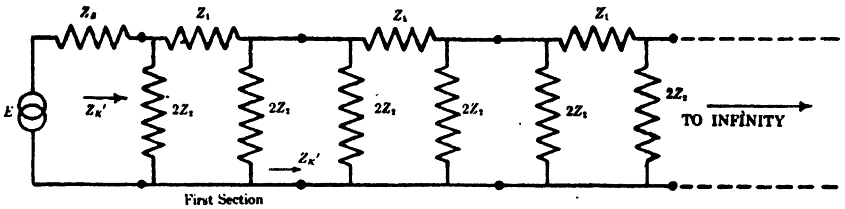

FIG. 4. Recurrent structure of infinite length regarded as made up of symmetrical Π networks.

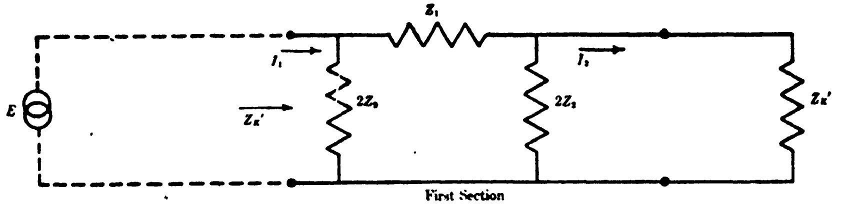

FIG. 5. Single Π network terminated in its mid-shunt iterative impedance Z_K'.

Whence

$$Z_K' = \frac{2Z_2\left[Z_1 + \dfrac{2Z_2Z_K'}{2Z_2 + Z_K'}\right]}{2Z_2 + Z_1 + \dfrac{2Z_2Z_K'}{2Z_2 + Z_K'}} \tag{8}$$

or solving for Z_K'

$$Z_K' = \sqrt{\frac{Z_1Z_2}{1 + \dfrac{1}{4}\dfrac{Z_1}{Z_2}}} \tag{9}$$

The ratio of the current entering the first section to the current leaving it is

$$\frac{I_1}{I_2} = \frac{2Z_1Z_2 + 4Z_2^2 + 4Z_2Z_K'\left(1 + \dfrac{Z_1}{4Z_2}\right)}{4Z_2^2} \tag{10}$$

Putting the value of Z_K' as found in (9) in the above equation,

$$\frac{I_1}{I_2} = \frac{\dfrac{Z_1}{2} + Z_2 + \sqrt{Z_1Z_2 + \dfrac{Z_1^2}{4}}}{Z_2} \tag{11}$$

which is identically the same as the expression for the current ratio given by equation (3). *The propagation constant, therefore, per unit length of a uniform line or per section of a line of recurrent structure will*

be the same whether the line or structure is considered to be made up of an infinite number of T networks or of an infinite number of Π networks. In other words, the ratio of the currents at adjacent mid-series points is the same as the ratio of the currents at adjacent mid-shunt points.

11.12 Properties of the Iterative Impedance and of the Propagation Constant.—Consider again one of the sections of the structure shown in Fig. 2—such as is represented in Fig. 6.

In this figure, the impedance Z_S at the 1–2 terminals when the 3–4 terminals are short-circuited is

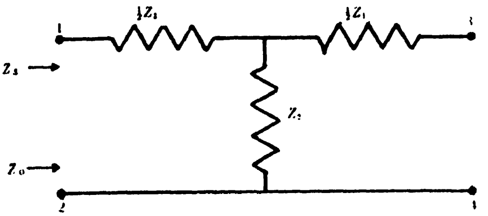

FIG. 6. Simple T network, representing any structure

$$Z_S = \frac{1}{2}Z_1 + \frac{\frac{1}{2}Z_1Z_2}{Z_2 + \frac{1}{2}Z_1} = \frac{\frac{Z_1^2}{2} + 2Z_1Z_2}{Z_1 + 2Z_2} \tag{12}$$

The impedance Z_O at the 1–2 terminals when the 3–4 terminals are open-circuited is

$$Z_O = \frac{1}{2}Z_1 + Z_2 = \frac{Z_1 + 2Z_2}{2} \tag{13}$$

Taking the square root of the product of (12) and (13),

$$Z = \sqrt{Z_OZ_S} = \sqrt{Z_1Z_2\left(1 + \frac{1}{4}\frac{Z_1}{Z_2}\right)} \tag{14}$$

which is exactly the same as the expression derived in (1) for the mid-series iterative impedance of an infinite number of T networks similar to the one shown in Fig. 6. Likewise, if one of the Π networks of the structure shown in Fig. 4 is considered, the short-circuit impedance (see Fig. 7) is

$$Z_S' = \frac{2Z_1Z_2}{Z_1 + 2Z_2} \tag{15}$$

and the open-circuit impedance is

$$Z_O' = \frac{2Z_2(Z_1 + 2Z_2)}{Z_1 + 4Z_2} \tag{16}$$

Taking the square root of the product of (15) and (16),

$$Z' = \sqrt{Z_O' Z_S'} = \sqrt{\frac{Z_1 Z_2}{1 + \frac{1}{4}\frac{Z_1}{Z_2}}} \tag{17}$$

which is the same expression as that derived in (9) for the mid-shunt iterative impedance of an infinite number of Π networks similar to the one shown in Fig. 7. *A property, then, of the iterative impedance of a recurrent symmetrical structure is its equality to the geometric mean of the open- and short-circuit impedances of the structure.*

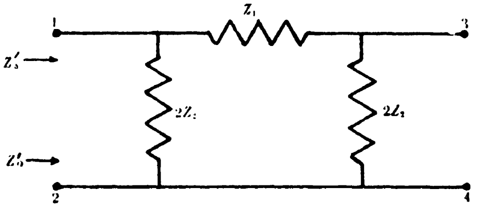

FIG. 7. Simple Π network, representing any structure.

In a similar manner, from either Fig. 6 or Fig. 7, it can be shown that the propagation constant, P, as expressed by equation (7) may also be written in the variant form

$$P = \tanh^{-1}\sqrt{\frac{Z_S}{Z_O}} = \tanh^{-1}\sqrt{\frac{Z_S'}{Z_O'}} \tag{18}$$

In other words, *a property of the propagation constant per section of a recurrent symmetrical structure is its equality to the hyperbolic anti-tangent of the square root of the ratio of the short-circuit impedance to the open-circuit impedance of the structure.*

11.2 Properties of Dissymmetrical Structures.—When we turn now to the discussion of the more general case of a dissymmetrical structure, we find ourselves confronted by several different methods of obtaining the characteristics of the structure. We may, for example, consider the structure to be one of an infinite number of similar structures and obtain the iterative impedances and propagation constant by a method similar to that used in Sect. 11.11, or we may evidently define the characteristics of the structure in terms of its open- and short-circuit impedances—or their equivalents—as is done in Sect. 11.12. Although in the special case of a symmetrical structure these two conceptions of the characteristics of the structure reduce to equivalent expressions, in the more general case of a dissymmetrical structure they lead to different formulae.

Since the nomenclature of the various terms involved has, in the past, been somewhat confused, it may be advisable at this point to state definitely the terminology and symbols which are used throughout this book. The *impedances and propagation constant corresponding to an infinite number of similar recurrent structures are designated as the iterative impedances,* Z_{K_1} *and* Z_{K_2}, *and as the propagation constant,* $P \equiv A + jB$, *respectively,* since in this concept the emphasis is placed upon wave propagation along an infinite line of recurrent structure. The line characteristics, as used in the alternate method of considering a passive network, are referred to as the *image impedances,* Z_{I_1} and Z_{I_2}, and the *image transfer constant,* $\Theta \equiv \alpha + j\beta$, the justification for these latter names being given in Sect. 11.22.[*] The real part, α, of the image transfer constant is called the *attenuation constant* and its imaginary part, β, is called the *phase constant.* In the case of a *symmetrical* structure, the propagation constant and the image transfer constant are identical, so that in such a structure the real part, A, of the propagation constant is likewise the *attenuation constant* and its imaginary part, B, is also the *phase constant.*

As is shown further on, the former concept is, in general, the more useful when dealing with recurrent structures, such as smooth and coil-loaded lines, etc., while the latter concept will usually be found more satisfactory in the consideration of such non-recurrent networks as composite wave filters, transformers, etc.

11.21 The Propagation Constant, P, and Iterative Impedances, Z_{K_1} and Z_{K_2}, of Recurrent Dissymmetrical Structures.—Let us consider now the general case of a structure composed of an infinite number of dissymmetrical T networks as shown in either Fig. 8A or Fig. 8B. Such a structure will obviously have two different iterative impedances, Z_{K_1} and Z_{K_2}. By a line of reasoning similar to that followed in Sect. 11.11, the first section of Fig. 8A may be represented as in Fig. 9. Writing the equation for the impedance Z_{K_1}

$$Z_{K_1} = Z_A + \frac{Z_C(Z_B + Z_{K_1})}{Z_B + Z_C + Z_{K_1}} \tag{19}$$

and solving for Z_{K_1},

$$Z_{K_1} = \sqrt{\frac{(Z_A + Z_B)(Z_A + Z_B + 4Z_C)}{4}} + \frac{Z_A - Z_B}{2} \tag{20}$$

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[*] *Iterative impedance* and *image impedance* seem to be much more satisfactory terms than loosely used terms such as *characteristic impedance, surge impedance,* etc.

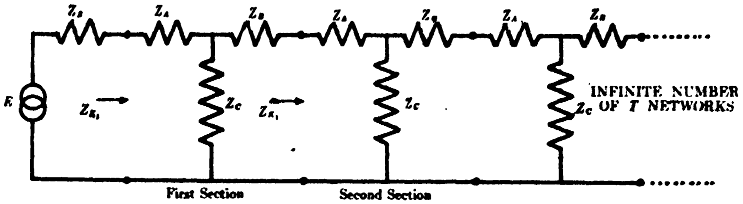

FIG. 8A.

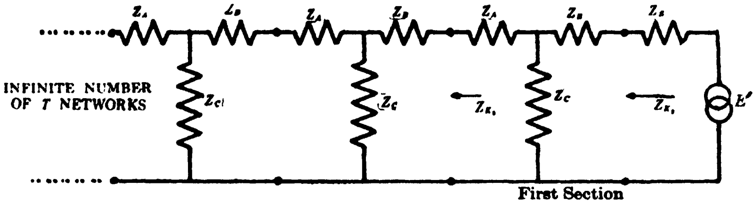

FIG. 8B.

FIGS. 8A and 8B. Recurrent structure of infinite length regarded as made up of dissymmetrical T networks (Z_A, Z_B and Z_C). Fig. 8A—E.M.F. applied at end nearest Z_A. Fig. 8B—E.M.F. applied at end nearest Z_B.

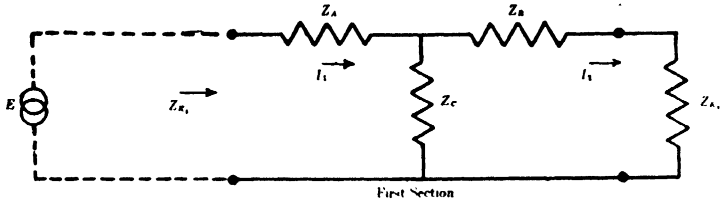

FIG. 9. Single dissymmetrical T network terminated in its iterative impedance Z_{K_1}.

Similarly, the first section of Fig. 8B may be represented as shown in Fig. 10. Writing the equation for the impedance Z_{K_2}

$$Z_{K_2} = Z_B + \frac{Z_C(Z_A + Z_{K_2})}{Z_A + Z_C + Z_{K_2}} \tag{21}$$

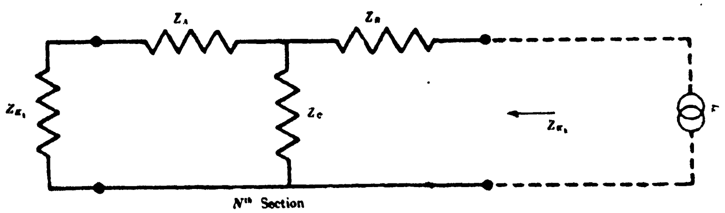

FIG. 10. Single dissymmetrical T network terminated in its iterative impedance Z_{K_2}.

and solving for Z_{K_2},

$$Z_{K_2} = \sqrt{\frac{(Z_A + Z_B)(Z_A + Z_B + 4Z_C)}{4}} - \frac{Z_A - Z_B}{2}. \tag{22}$$

Equations (20) *and* (22), *are then, the expressions for the two iterative impedances of a structure composed of an infinite number of similar recurrent dissymmetrical T networks.* If, in these expressions, Z_A is equal to Z_B, they both reduce to the form of the iterative impedance of an infinite number of symmetrical recurrent T networks as given by equation (1).

Consider now the propagation constant of the first section as shown in Fig. 9. By Kirchhoff's Laws

$$\frac{I_1}{I_2} = \frac{Z_B + Z_C + Z_{K_1}}{Z_C} \tag{23}$$

Taking the natural logarithm of this expression and substituting for Z_{K_1} its value as given by (20), it can be shown from formula (6) that

$$P \equiv \log_e \frac{I_1}{I_2} = \cosh^{-1}\left(\frac{Z_A + Z_B + 2Z_C}{2Z_C}\right) \tag{24}$$

Equation (24), *then, expresses the propagation constant per section of a structure consisting of recurrent dissymmetrical T networks.* Note also that if $Z_A = Z_B$, equation (24) reduces to the form of the formula for the propagation constant for recurrent symmetrical T networks as given in the second expression of equation (7).

11.22 The Image Transfer Constant, Θ, and the Image Impedances, Z_{I_1} and Z_{I_2}, of Any Dissymmetrical Structure.—Let us now consider the characteristics of any dissymmetrical structure in the light of the second concept previously mentioned. Since any structure can, in general, be represented by a dissymmetrical T network, we shall confine our attentions to the discussion of the characteristics of the T network shown in Fig. 11. Now if Z_{S_1} is the input impedance at the 1–2 terminals when the 3–4 terminals are short-circuited, and Z_{O_1} is the input impedance at the 1–2 terminals when the 3–4 terminals are open-circuited—with corresponding definitions for Z_S and Z_{O_2}—the equations for the two

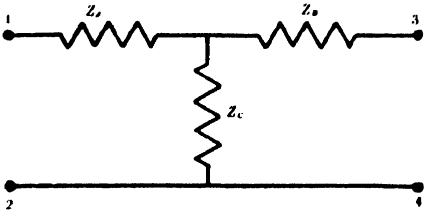

FIG. 11. Dissymmetrical T network.

image impedances of the network are evidently

$$Z_{I_1} = \sqrt{Z_{O_1}Z_{S_1}} = \sqrt{\frac{(Z_A + Z_C)(Z_AZ_B + Z_AZ_C + Z_BZ_C)}{(Z_B + Z_C)}} \tag{25}$$

$$Z_{I_2} = \sqrt{Z_{O_2}Z_{S_2}} = \sqrt{\frac{(Z_B + Z_C)(Z_AZ_B + Z_AZ_C + Z_BZ_C)}{(Z_A + Z_C)}} \tag{26}$$

As already demonstrated in Sect. 7.5—equations (86) and (87)—the impedances given by equations (25) and (26) are also the impedances which will terminate the network without reflection losses. We can, therefore, define *the image impedances,* Z_{I_1} *and* Z_{I_2}, *of any network having two input and two output terminals as those impedances which will terminate the network in such a way that at either junction the impedance in either direction is the same.* In other words, with Z_{I_1} connected to the terminals 1–2 (see Fig. 11) and Z_{I_2} connected to the terminals 3–4, each will face an equal impedance or its own *image* as illustrated in Fig. 12. Consequently, the term *image impedances* seems to be appropriate for Z_{I_1} and Z_{I_2}.

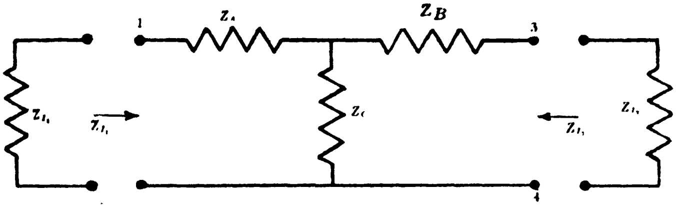

FIG. 12. Dissymmetrical *T* network connected to its image impedances Z_{I_1} and Z_{I_2}.

Equations (25) *and* (26) *are, then, the expressions for the image impedances of any dissymmetrical structure in terms of the impedances of the arms of its equivalent T network.* It will be noted also that when the structure is symmetrical ($Z_A = Z_B$), (25) and (26) both reduce to the same form of expression as that for the iterative impedance of a symmetrical structure, as given by equation (1). Hence, *in the symmetrical case, the iterative and the image impedances are identical.*

Referring again to Fig. 12, the image transfer constant Θ of the structure (see Sect. 7.52) is

$$\begin{aligned} \Theta = \tanh^{-1}\sqrt{\frac{Z_{S_1}}{Z_{O_1}}} &= \tanh^{-1}\sqrt{\frac{Z_{S_2}}{Z_{O_2}}} \\ &= \tanh^{-1}\sqrt{\frac{Z_AZ_B + Z_AZ_C + Z_BZ_C}{(Z_A + Z_C)(Z_B + Z_C)}} \qquad (27) \\ &= \cosh^{-1}\frac{\sqrt{(Z_A + Z_C)(Z_B + Z_C)}}{Z_C} \end{aligned}$$

Equation (27), *then, expresses the image transfer constant of any dissymmetrical structure in terms of the impedances of the arms of its equivalent T network.* In the symmetrical case, ($Z_A = Z_B$), it is seen that the third and fourth expressions in equation (27) reduce to the fourth and second expressions, respectively, in equation (7) for the propagation constant of a symmetrical structure. *Hence, for any symmetrical network, the image transfer constant* Θ *is the same as the propagation constant* P.

The discussion in Sects. 11.21 and 11.22 has been based on the assumption that the recurrent structure is made up of dissymmetrical T networks. The relationships of the iterative and image impedances, as well as that of the propagation constant and the image transfer constant could, however, have been developed equally well by employing dissymmetrical Π networks, or in fact any other form of passive network.

In passing, it might be pointed out that a very simple relation exists between the iterative and the image impedances of a dissymmetrical structure, namely, that the geometric mean of the two iterative impedances is equal to the geometric mean of the two image impedances, or

$$\sqrt{Z_{I_1} Z_{I_2}} = \sqrt{Z_{K_1} Z_{K_2}}$$

This geometric mean is called by Kennelly the *geomean surge impedance.*

11.23 Numerical Example of the Various Properties of a Dissymmetrical Structure.—In order to illustrate the difference between the two methods outlined above of dealing with a dissymmetrical structure, a simple numerical example will be taken and values obtained for its image and iterative impedances and its image transfer and propagation constants. It will be assumed that the impedances of the arms of the dissymmetrical T network shown in Fig. 11 are pure resistances and have the following values:

$$\begin{aligned} Z_A &= 1 \text{ ohm} \\ Z_B &= 100 \text{ ohms} \\ Z_C &= 100 \text{ ohms} \end{aligned}$$

From equations (20), (22), (24), (25), (26) and (27) the following values have been calculated:

$$\begin{aligned} Z_{K_1} &= 63.0 \text{ ohms} \\ Z_{K_2} &= 162.0 \text{ ohms} \\ P &= 0.97 \text{ napier} \end{aligned}$$

$$\begin{aligned} Z_{I_1} &= 71.8 \text{ ohms} \\ Z_{I_2} &= 142.1 \text{ ohms} \\ \Theta &= 0.89 \text{ napier} \end{aligned}$$

As mentioned above, the geometric means of the image and iterative impedances are equal to each other and, in the case under consideration, have the value:

$$\sqrt{(63)(162)} = \sqrt{(71.8)(142.1)} = 101 \text{ ohms}$$

11.3 Equivalent *T* Network of a Symmetrical Structure—Expressed in Terms of Its Propagation Constant and Its Iterative Impedance.—

From equation (7) the propagation constant P per section of the structure shown in Fig. 1 is

$$P = \sinh^{-1} \frac{\sqrt{Z_1 Z_2 \left(1 + \frac{1}{4}\frac{Z_1}{Z_2}\right)}}{Z_2} \tag{28}$$

Combining this with equation (1),

$$P = \sinh^{-1} \frac{Z_K}{Z_2} \quad \text{or} \quad Z_2 = \frac{Z_K}{\sinh P} \tag{29}$$

Similarly, from equation (7)

$$P = 2 \tanh^{-1} \frac{1}{\sqrt{1 + 4\frac{Z_2}{Z_1}}} \quad \text{or} \quad \tanh\frac{P}{2} = \frac{1}{\sqrt{1 + 4\frac{Z_2}{Z_1}}} \tag{30}$$

Whence

$$\frac{\tanh\frac{P}{2}}{Z_1} = \frac{1}{2\sqrt{Z_1 Z_2 \left[1 + \frac{1}{4}\frac{Z_1}{Z_2}\right]}} \tag{31}$$

Combining this last expression with equation (1) gives

$$\frac{\tanh\frac{P}{2}}{Z_1} = \frac{1}{2Z_K} \quad \text{or} \quad \frac{Z_1}{2} = Z_K \tanh\frac{P}{2} \tag{32}$$

From equations (29) and (32) it is apparent that the *equivalent T network* of a symmetrical structure having a propagation constant P and an iterative impedance Z_K is the network shown in Fig. 13.

If an e.m.f. E is applied to one end of such a structure, through an impedance Z_S, and if the opposite terminals are closed through an impedance Z_R—as is shown in Fig. 14—it is evident from an inspection of the structure that both the input and the output currents I_S and I_R depend solely upon the values of the terminal impedances Z_S and Z_R, and the values of the iterative impedance Z_K and the propagation constant P of the structure. Consequently, if the iterative impedance Z_K and the total propagation constant P of any symmetrical structure (such as a line, for example), are known the actual structure can be replaced by its equivalent T network, as indicated in Fig. 14, without affecting the currents in the external circuit (external to the points 1–2 and 3–4).

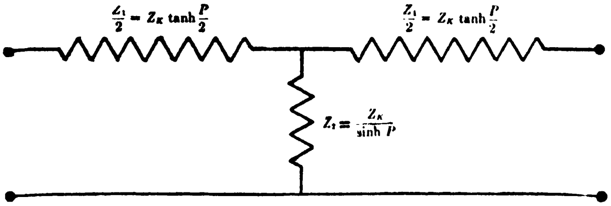
$\frac{Z_1}{2} = Z_K \tanh\frac{P}{2}$
$\frac{Z_1}{2} = Z_K \tanh\frac{P}{2}$
$Z_3 = \frac{Z_K}{\sinh P}$

FIG. 13. Equivalent T network representing any symmetrical structure having an iterative impedance Z_K and a propagation constant P.

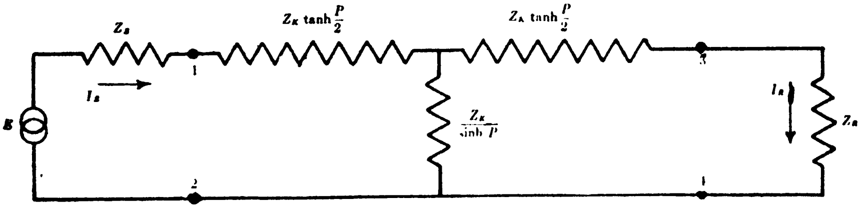

FIG. 14. Equivalent T network of Fig. 13 connected between generalized terminal impedances Z_R and Z_S.

11.4 Equivalent T Network of Any Dissymmetrical Structure—Expressed in Terms of Its Propagation Constant and Its Iterative Impedances.—Let us consider now the general case of the equivalent T network of any dissymmetrical structure and derive expressions for the various arms in terms of the iterative impedances, Z_{K_1} and Z_{K_2}, and the propagation constant P. As mentioned above, this method of representing a dissymmetrical structure will not generally be found as useful as the method described in Sect. 11.5.

From formulae (20), (22) and (24), it is known that the expressions for Z_{K_1}, Z_{K_2} and P of the structure shown in Fig. 15—in terms of the

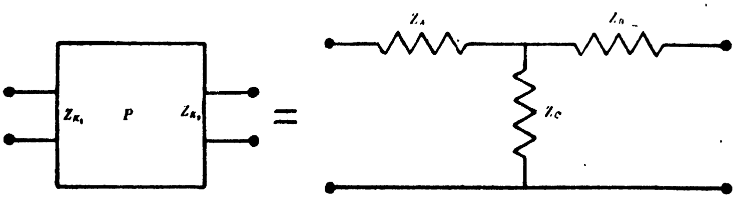

FIG. 15. Dissymmetrical structure having iterative impedances Z_{K_1} and Z_{K_2} and propagation constant P with its equivalent dissymmetrical T network.

various arms of its equivalent *T* network,—are

$$Z_{K_1} = \sqrt{\frac{(Z_A + Z_B)(Z_A + Z_B + 4Z_C)}{4}} + \frac{Z_A - Z_B}{2} \tag{33}$$

$$Z_{K_2} = \sqrt{\frac{(Z_A + Z_B)(Z_A + Z_B + 4Z_C)}{4}} - \frac{Z_A - Z_B}{2} \tag{34}$$

$$P = \cosh^{-1}\left(\frac{Z_A + Z_B + 2Z_C}{2Z_C}\right) \tag{35}$$

Solving equations (33), (34) and (35) simultaneously for Z_A, Z_B and Z_C, the following relations are obtained:

$$Z_A = \frac{Z_{K_1} + Z_{K_2}}{2}\tanh\frac{P}{2} + \frac{Z_{K_1} - Z_{K_2}}{2} \tag{36}$$

$$Z_B = \frac{Z_{K_1} + Z_{K_2}}{2}\tanh\frac{P}{2} - \frac{Z_{K_1} - Z_{K_2}}{2} \tag{37}$$

$$Z_C = \frac{Z_{K_1} + Z_{K_2}}{2\sinh P} \tag{38}$$

The equivalent *T* network, therefore, of any dissymmetrical structure expressed in terms of its propagation constant and its iterative impedances is as shown in Fig. 16.

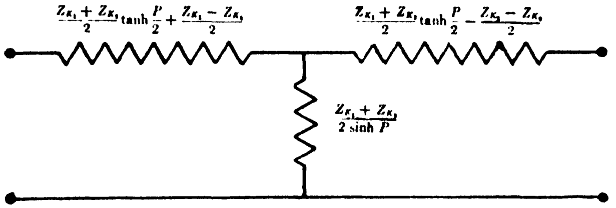
$\frac{Z_{K_1}+Z_{K_2}}{2}\tanh\frac{P}{2}+\frac{Z_{K_1}-Z_{K_2}}{2}$
$\frac{Z_{K_1}+Z_{K_2}}{2}\tanh\frac{P}{2}-\frac{Z_{K_1}-Z_{K_2}}{2}$
$\frac{Z_{K_1}+Z_{K_2}}{2\sinh P}$

FIG. 16. Equivalent *T* network representing any dissymmetrical structure whose iterative impedances are Z_{K_1} and Z_{K_2} and whose propagation constant is *P*.

11.5 Equivalent *T* Network of Any Dissymmetrical Structure—Expressed in Terms of Its Image Transfer Constant and Its Image Impedances.—By a line of reasoning analogous to that followed in Sect. 11.4, the equivalent *T* network can be derived for a dissymmetrical structure in terms of its image impedances, Z_{I_1} and Z_{I_2}, and its image transfer constant Θ. This will be shown, in Chapter XVII, to be the most useful and satisfactory method of representing any dissymmetrical passive network, such as composite wave filters, etc.

Referring to Fig. 17, it is known that the three following equations,

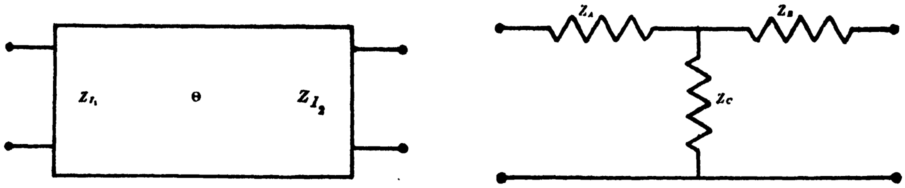

FIG. 17. Dissymmetrical structure having image impedances Z_{I_1} and Z_{I_2} and an image transfer constant Θ together with its equivalent T network.

which are equations (25), (26) and (27) above, apply

$$Z_{I_1} = \sqrt{\frac{(Z_A + Z_C)(Z_A Z_B + Z_A Z_C + Z_B Z_C)}{Z_B + Z_C}} \tag{39}$$

$$Z_{I_2} = \sqrt{\frac{(Z_B + Z_C)(Z_A Z_B + Z_A Z_C + Z_B Z_C)}{Z_A + Z_C}} \tag{40}$$

$$\Theta = \tanh^{-1}\sqrt{\frac{Z_A Z_B + Z_A Z_C + Z_B Z_C}{(Z_A + Z_C)(Z_B + Z_C)}} \tag{41}$$

Solving equations (39), (40) and (41) simultaneously for Z_A, Z_B and Z_C, the following relations are obtained:

$$Z_A = \frac{Z_{I_1}}{\tanh\Theta} - \frac{\sqrt{Z_{I_1}Z_{I_2}}}{\sinh\Theta} \tag{42}$$

$$Z_B = \frac{Z_{I_2}}{\tanh\Theta} - \frac{\sqrt{Z_{I_1}Z_{I_2}}}{\sinh\Theta} \tag{43}$$

$$Z_C = \frac{\sqrt{Z_{I_1}Z_{I_2}}}{\sinh\Theta} \tag{44}$$

Therefore, in so far as its external action is concerned, any dissymmetrical structure may be replaced by a T network in which the various arms have the values shown in Fig. 18.

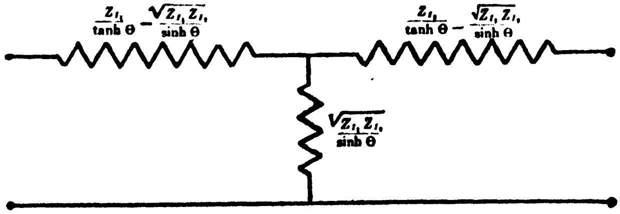
$\frac{Z_{I_1}}{\tanh\Theta} - \frac{\sqrt{Z_{I_1}Z_{I_2}}}{\sinh\Theta}$
$\frac{Z_{I_2}}{\tanh\Theta} - \frac{\sqrt{Z_{I_1}Z_{I_2}}}{\sinh\Theta}$
$\frac{\sqrt{Z_{I_1}Z_{I_2}}}{\sinh\Theta}$

FIG. 18. Equivalent T network representing any dissymmetrical structure whose image impedances are Z_{I_1} and Z_{I_2} and whose image transfer constant is Θ.

Note that in a symmetrical structure ($Z_{I_1} = Z_{I_2} = Z_I = Z_{K_1} = Z_{K_2} = Z_K$); Figs. 16 and 18 then reduce, as they should, to the same equivalent T network, namely, that of the symmetrical structure represented in Fig. 13.

11.6 Impedance of Any Structure Terminated in an Impedance, Z_R, and Expressed in Terms of Z_K and P or Z_I and Θ.—Consider the equivalent T network of the dissymmetrical structure shown in Fig. 19

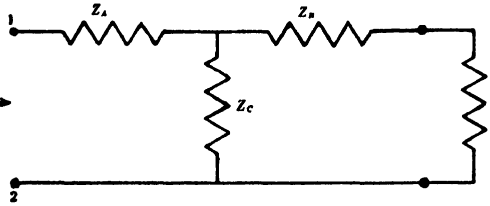

FIG. 19. Dissymmetrical T network terminated in an impedance Z_R

where Z_A, Z_B and Z_C have the values given by equations (42), (43) and (44). The impedance Z measured at the terminals 1–2 is

$$Z = Z_A + \frac{Z_C(Z_B + Z_R)}{Z_B + Z_C + Z_R} \tag{45}$$

Substituting the values of Z_A, Z_B and Z_C, equation (45) reduces to

$$Z = Z_{I_1}\left[\frac{Z_R + Z_{I_2} \tanh \Theta}{Z_{I_2} + Z_R \tanh \Theta}\right] \tag{46}$$

In a similar manner, by substituting in equation (45) the values given in (36), (37) and (38), an expression can be derived for the impedance in terms of the propagation constant and the iterative impedances of the structure, namely,

$$Z = \frac{Z_R(Z_{K_1} + Z_{K_2}) + (2Z_{K_1}Z_{K_2} + Z_R Z_{K_1} - Z_R Z_{K_2}) \tanh P}{(Z_{K_1} + Z_{K_2}) + (2Z_R - Z_{K_1} + Z_{K_2}) \tanh P} \tag{47}$$

In general, (47) will not be found to be as useful an expression as that given in (46).

In the case of a symmetrical structure, equations (46) and (47) both reduce to the familiar form

$$Z = Z_I\left[\frac{Z_R + Z_I \tanh \Theta}{Z_I + Z_R \tanh \Theta}\right] = Z_K\left[\frac{Z_R + Z_K \tanh P}{Z_K + Z_R \tanh P}\right] \tag{48}$$

Equation (46), *therefore, gives the impedance Z of any dissymmetrical structure having image impedances Z_{I_1} and Z_{I_2}, a total image transfer constant Θ and terminated in any impedance Z_R. It should be noted that in the case of a symmetrical structure, when $\Theta = P = 0$ (in which case*

$\tanh\Theta = \tanh P = 0$) the impedance Z is equal to that of the terminal impedance Z_R, while if the attenuation is infinite (in which case $\tanh\Theta = \tanh P = 1$) the impedance Z is equal to the iterative impedance Z_K or to the image impedance Z_I of the structure and is independent of the value of Z_R.

11.7 Current at the Receiving End of Any Structure, Expressed in Terms of Z_I and Θ or Z_K and P.—From the relations already derived in Sect. 11.5 it is known that the structures shown in Fig. 20 are

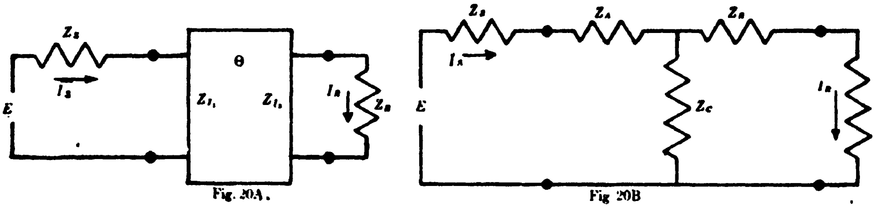

Fig. 20A. Fig 20B

FIGS. 20A and 20B. Dissymmetrical structures connected to terminal impedances Z_S and Z_R. Fig. 20A—Structure having image impedances Z_{I_1} and Z_{I_2} and an image transfer constant Θ. Fig. 20B—Equivalent T network of the structure shown in Fig. 20A.

externally equivalent provided

$$Z_A = \frac{Z_{I_1}}{\tanh\Theta} - \frac{\sqrt{Z_{I_1}Z_{I_2}}}{\sinh\Theta} \tag{49}$$

$$Z_B = \frac{Z_{I_2}}{\tanh\Theta} - \frac{\sqrt{Z_{I_1}Z_{I_2}}}{\sinh\Theta} \tag{50}$$

$$Z_C = \frac{\sqrt{Z_{I_1}Z_{I_2}}}{\sinh\Theta} \tag{51}$$

From Kirchhoff's Laws it can be shown that the received current I_R in Fig. 20B is

$$I_R = \frac{EZ_C}{(Z_S + Z_A)(Z_B + Z_C) + Z_R(Z_A + Z_C) + Z_BZ_C + Z_SZ_R} \tag{52}$$

Substituting the values of Z_A, Z_B and Z_C, as given in equations (49), (50) and (51), in equation (52),

$$I_R = \frac{E\sqrt{Z_{I_1}Z_{I_2}}}{(Z_{I_1}Z_{I_2} + Z_SZ_R)\sinh\Theta + (Z_{I_1}Z_R + Z_{I_2}Z_S)\cosh\Theta} \tag{53}$$

Multiplying this latter expression by certain factors for a better physical interpretation, equation (53) can be expressed in the following form

$$I_R = \frac{E}{Z_S + Z_R} \times \frac{Z_S + Z_R}{\sqrt{4Z_S Z_R}} \times \frac{\sqrt{4Z_{I_1} Z_S}}{Z_{I_1} + Z_S} \times \frac{\sqrt{4Z_{I_2} Z_R}}{Z_{I_2} + Z_R}$$

$$\times \epsilon^{-\Theta} \times \frac{1}{1 - \dfrac{Z_{I_2} - Z_R}{Z_{I_2} + Z_R} \times \dfrac{Z_{I_1} - Z_S}{Z_{I_1} + Z_S} \times \epsilon^{-2\Theta}} \qquad (54)$$

In equation (54), the first factor gives the current which would have existed if no structure had been inserted between Z_S and Z_R. The next three factors in this equation are seen to be all of the same general type except that the first of the three is similar to the reciprocal of the other two factors. These two latter factors have been called *reflection factors* and determine the *reflection loss* which is said to exist between the two impedances involved. The fifth factor is the *transfer factor* and gives the reduction in current which is due strictly to attenuation while the the last factor has been called the *interaction factor*. The value of the reflection factors is evidently a function simply of the *ratio* of the impedances involved while the absolute value of the transfer factor is $\epsilon^{-\alpha}$, in which α is the attenuation constant of the structure. The value of this factor may, therefore, evidently be read at once from exponential tables as soon as the value of α is known. The value of the interaction factor, like that of the reflection factors, is seen to be unity either when $Z_{I_2} = Z_R$ or when $Z_{I_1} = Z_S$. The interaction factor also becomes unity if the value of Θ is infinitely large.

In a like manner, by using the values of Z_A, Z_B and Z_C given by equations (36), (37) and (38), the received current can be expressed in terms of the total propagation constant P and the iterative impedances Z_{K_1} and Z_{K_2} of a structure, namely,

$$I_R = \frac{E}{Z_S + Z_R} \times \frac{Z_S + Z_R}{\sqrt{4Z_S Z_R}} \times \frac{Z_{K_1} + Z_{K_2}}{\sqrt{4Z_{K_1} Z_{K_2}}} \times \frac{\sqrt{4Z_{K_1} Z_S}}{Z_{K_1} + Z_S}$$

$$\times \frac{\sqrt{4Z_{K_2} Z_R}}{Z_{K_2} + Z_R} \times \epsilon^{-P} \times \frac{1}{1 - \dfrac{Z_{K_2} - Z_S}{Z_K + Z_S} \times \dfrac{Z_{K_1} - Z_R}{Z_{K_2} + Z_R} \times \epsilon^{-2P}} \qquad (55)$$

As already mentioned, in the case of a dissymmetrical structure this latter expression will not generally be found as useful as equation (54) in which the received current is expressed in terms of the image transfer constant, Θ, and the image impedances, Z_{I_1} and Z_{I_2}, of the structure.

If the structure is symmetrical, (54) reduces to the form

$$I_R = \frac{E}{Z_S + Z_R} \times \frac{Z_S + Z_R}{\sqrt{4Z_S Z_R}} \times \frac{\sqrt{4Z_I Z_S}}{Z_I + Z_S} \times \frac{\sqrt{4Z_I Z_R}}{Z_I + Z_R}$$
$$\times \epsilon^{-\Theta} \times \frac{1}{1 - \frac{Z_I - Z_R}{Z_I + Z_R} \times \frac{Z_I - Z_S}{Z_I + Z_S} \times \epsilon^{-2\Theta}} \qquad (56)$$

Similarly, (55) will reduce to the same expression if we substitute in (56) for Z_I its equivalent, Z_K, and for Θ its equivalent, P.

If, furthermore, $Z_S = Z_R$, equation (56) reduces still further to the form

$$I_R = \frac{E}{2Z_R} \times \frac{4Z_I Z_R}{(Z_I + Z_R)^2} \times \epsilon^{-\Theta} \times \frac{1}{1 - \left(\frac{Z_I - Z_R}{Z_I + Z_R}\right)^2 \times \epsilon^{-2\Theta}} \qquad (57)$$

or

$$I_R = \frac{E}{2Z_R} \times \frac{4Z_K Z_R}{(Z_K + Z_R)^2} \times \epsilon^{-P} \times \frac{1}{1 - \left(\frac{Z_K - Z_R}{Z_K + Z_R}\right)^2 \times \epsilon^{-2P}} \qquad (58)$$

11.8 Transmission Loss Caused by Any Structure, Expressed in Terms of Z_I and Θ or Z_K and P.—As already mentioned in Chapter IX, a method of calculating the transmission loss of any structure that is frequently more convenient than the method indicated in Fig. 1 of Chapter IX is by means of a knowledge of the image impedances Z_{I_1} and Z_{I_2} and of the image transfer constant Θ of the structure. This method is particularly applicable in the case of structures, such as composite wave filters, which ordinarily consist of a number of sections so joined together that they have like image impedances at their respective junctions. As will be shown in Sects. 17.41 and 17.42, the image impedances Z_{I_1} and Z_{I_2} of such a structure will be identical with the image impedances respectively of the first and last half sections, while the total image transfer constant Θ will be the sum of the image transfer constants of the various sections.

If we insert a structure of image impedances Z_{I_1} and Z_{I_2} and total image transfer constant Θ between terminal impedances Z_S and Z_R, the received current is given by equation (54). If the structure whose constants are Z_{I_1}, Z_{I_2} and Θ had not been inserted, as above assumed, the received current I_R' would have been

$$I_R' = \frac{E}{Z_S + Z_R} \qquad (59)$$

It is, therefore, evident that the ratio of the received currents in the two cases is

$$\frac{I_R}{I_R'} = \frac{Z_S + Z_R}{\sqrt{4Z_SZ_R}} \times \frac{\sqrt{4Z_{I_1}Z_S}}{Z_{I_1} + Z_S} \times \frac{\sqrt{4Z_{I_2}Z_R}}{Z_{I_2} + Z_R} \times \epsilon^{-\Theta}$$

$$\times \frac{1}{1 - \frac{Z_{I_2} - Z_R}{Z_{I_2} + Z_R} \times \frac{Z_{I_1} - Z_S}{Z_{I_1} + Z_S} \times \epsilon^{-2\Theta}} \qquad (60)$$

Using equation (55), a somewhat similar expression for the current ratio may be obtained in terms of the terminal impedances, the iterative impedances Z_{K_1} and Z_{K_2} of the structure and its total propagation constant P, namely,

$$\frac{I_R}{I_R'} = \frac{Z_S + Z_R}{\sqrt{4Z_SZ_R}} \times \frac{Z_{K_1} + Z_{K_2}}{\sqrt{4Z_{K_1}Z_{K_2}}} \times \frac{\sqrt{4Z_{K_1}Z_S}}{Z_{K_1} + Z_S} \times \frac{\sqrt{4Z_{K_2}Z_R}}{Z_{K_2} + Z_R}$$

$$\times \epsilon^{-P} \times \frac{1}{1 - \frac{Z_{K_2} - Z_S}{Z_{K_1} + Z_S} \times \frac{Z_{K_1} - Z_R}{Z_{K_2} + Z_R} \times \epsilon^{-2P}} \qquad (61)$$

In general, when the structure is dissymmetrical, this latter expression will not be found as useful as the one given in (60) involving the image impedances and the image transfer constant.

By referring to equation (60), it will be seen that, if the structure is symmetrical, the two image impedances will be alike ($Z_{I_1} = Z_{I_2} = Z_I$) and the expression will assume the form of

$$\frac{I_R}{I_R'} = \frac{Z_S + Z_R}{\sqrt{4Z_SZ_R}} \times \frac{\sqrt{4Z_IZ_S}}{Z_I + Z_S} \times \frac{\sqrt{4Z_IZ_R}}{Z_I + Z_R} \times \epsilon^{-\Theta}$$

$$\times \frac{1}{1 - \frac{Z_I - Z_R}{Z_I + Z_R} \times \frac{Z_I - Z_S}{Z_I + Z_S} \times \epsilon^{-2\Theta}} \qquad (62)$$

If the structure is symmetrical and, furthermore, if the sending end impedance Z_S is equal to the receiving end impedance Z_R, (62) becomes

$$\frac{I_R}{I_R'} = \epsilon^{-\Theta} \times \frac{4Z_IZ_R}{(Z_I + Z_R)^2} \times \frac{1}{1 - \left(\frac{Z_I - Z_R}{Z_I + Z_R}\right)^2 \times \epsilon^{-2\Theta}} \qquad (63)$$

Since in a symmetrical structure the iterative impedance Z_K is identical with the image impedance Z_I and the propagation constant P is identical with the image transfer constant Θ, equation (61) will, in the symmetrical case, reduce to equations (62) and (63) provided that in the latter expressions Z_I and Θ are replaced respectively by Z_K and P.

It is apparent, then, that (60) makes it possible to calculate rigorously the transmission loss caused by any structure whose image impedances and total image transfer constant are known. It will also be noted that in the symmetrical case if Z_I (or Z_K) $= Z_S = Z_R$, the transmission loss is equal simply to the attenuation loss of the structure. The interaction factor is also seen to approach unity when the real part, α, of the total image transfer constant, Θ, of the structure is large (say over one attenuation unit or napier); so that under these conditions the transmission loss is determined simply by the reflection factors and the value of α or it is determined by the ratio of the image impedance of the structure to the value of the terminal impedances and by the total attenuation constant of the structure.

The above method of determining the transmission loss of a structure, by means of its image impedances and its image transfer constant, is particularly applicable in the case of wave filters or similar structures where the image impedances are, in general, very closely equal to that of the terminal impedances throughout the frequency range in which the attenuation constant is relatively small, since under these conditions the attenuation constant is evidently the only factor that needs to be considered. On the other hand, in the attenuated range of such structures the interaction factor approaches unity and the transmission loss is determined simply by the attenuation constant and by the various reflection factors.

CHAPTER XII

TRANSMISSION CHARACTERISTICS OF TELEPHONE LINES

12.0 When an alternating voltage is impressed at one end of a line, part of the power entering the line is dissipated as heat, part of it is stored in the inductance and capacity of the circuit, and part of it is transmitted to the apparatus at the distant end.

In view of the fact that the frequency used in power transmission lines is so low, the ratio of the geographical length to the length of the electrical wave transmitted is small; that is to say, power lines are *electrically short.* In such lines a large part of the power entering the line can be delivered at the receiving end. The power input to the line, therefore, at a given voltage depends largely upon the power taken by the receiving device or load. In other words, the effective circuit impedance can be increased by increasing the impedance of the terminal apparatus. This is common practice in power transmission systems, the impedance of terminal apparatus as measured from the line being greatly increased by the use of high ratio transformers.

In the ordinary telephone line, however, transmitting currents at high frequencies, the ratio of the geographical length to the length of the electrical wave transmitted is large; that is, the lines are *electrically long.* Most of the power imparted to the line is dissipated in the line and it is possible to extract only a small fraction of the power from the receiving end of the line. Under these conditions it is evident that the power input to the line at a given voltage is very nearly independent of the power taken by the receiving apparatus. That is, the effective impedance of a long line is affected but very slightly by the impedance of the terminal apparatus and is practically fixed by the constants of the line itself.

It was shown in Chapter XI that the current entering or leaving any symmetrical structure, such as a line, may be regarded simply as a function of the iterative impedance and the total propagation constant of the line. Let us, therefore, consider what are the relations between the propagation constant and the iterative impedance of a line and its linear constants.

12.1 The Propagation Constant and Iterative Impedance of a Circuit Having Uniformly Distributed Constants.—Consider a structure such as is shown in Fig. 1. If the total number of sections in such a structure is $1/\Delta$, it is evident that the total series resistance of the structure will be R ohms, the total series inductance L henries, the total shunt conductance G mhos and the total shunt capacity C farads.

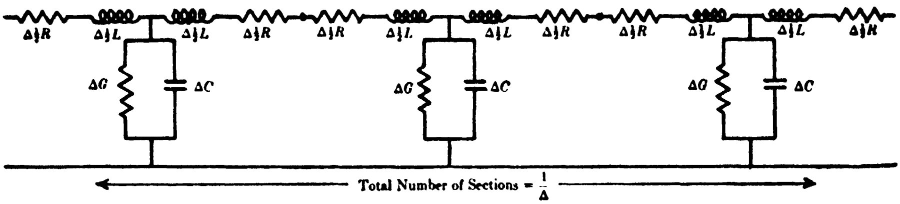

FIG. 1. Recurrent structure having series inductance and resistance, and shunt capacity and leakance.

From equation (1) in Chapter XI, the iterative impedance is

$$Z_K = \sqrt{\frac{\Delta(R + jL\omega)}{\Delta(G + jC\omega)}\left[1 + \frac{1}{4}\Delta(R + jL\omega) \times \Delta(G + jC\omega)\right]} \qquad (1)$$

Referring to equation (7) in Chapter XI and the paragraph below it, the total propagation constant, P, of the above structure is evidently

$$P = \frac{1}{\Delta} \times 2 \sinh^{-1}\frac{1}{2}\sqrt{\Delta(R + jL\omega) \times \Delta(G + jC\omega)} \qquad (2)$$

As Δ approaches zero, the structure (see Fig. 1) evidently approaches that of a line with uniformly distributed constants—the total values of which are R, L, G and C. Under the above assumption regarding Δ, (1) reduces to

$$Z_K = \sqrt{\frac{R + jL\omega}{G + jC\omega}} \qquad (3)$$

Also since

$$\epsilon^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$$

and $\sinh x = \frac{1}{2}(\epsilon^x - \epsilon^{-x})$, it can be shown that for small values of x, the value of $\sinh x$ approaches the value of x. Hence, under the above assumption regarding Δ, (2) becomes

$$P = \sqrt{(R + jL\omega)(G + jC\omega)} \qquad (4)$$

where P is the total propagation constant of a line having uniformly distributed constants—the total values of the constants being R, L, G and C. Similarly, (3) gives the iterative impedance of such a line. Since R, L, G and C are all proportional to the length of a line, it is apparent that *the propagation constant P is directly proportional to the length of the line while the iterative impedance Z_K is entirely independent of the circuit length.*

It will be noted from (3) and (4) that the product of P and Z_K gives $R + jL\omega$ and that the ratio of P to Z_K produces $G + jC\omega$, so that the total values of the primary constants (R, L, G and C) of a line having smoothly distributed constants are obtainable from a knowledge of the values of P and Z_K at any given frequency.

12.2 Simplified Expressions for the Attenuation Constant of a Uniform Line Having Smoothly Distributed Constants.—Although it is easy in any specific case to compute, from formula (4), the values of the attenuation constant A and the phase constant B from the fundamental constants R, L, G, C and ω, it is nevertheless possible to transform this formula in various ways so that a clearer idea may be obtained as to how A and B individually depend upon these constants.

From formula (4)

$$P \equiv A + jB = \sqrt{(R + jL\omega)(G + jC\omega)} \tag{5}$$

where P is the propagation constant, per unit length, of a uniform line having smoothly distributed constants R, L, G and C per unit length.

Squaring equation (5) and equating real parts to real and imaginary parts to imaginary,

$$A^2 - B^2 = RG - LC\omega^2 \tag{6}$$

and

$$2AB = RC\omega + GL\omega \tag{7}$$

Squaring equation (6),

$$A^4 - 2A^2B^2 + B^4 = R^2G^2 - 2RGLC\omega^2 + L^2C^2\omega^4 \tag{8}$$

Squaring equation (7),

$$4A^2B^2 = R^2C^2\omega^2 + 2RGLC\omega^2 + G^2L^2\omega^2 \tag{9}$$

Adding equations (8) and (9),

$$A^4 + 2A^2B^2 + B^4 = R^2(G^2 + C^2\omega^2) + L^2\omega^2(G^2 + C^2\omega^2) \tag{10}$$

Taking the square root,

$$A^2 + B^2 = \sqrt{(R^2 + L^2\omega^2)(G^2 + C^2\omega^2)} \tag{11}$$

Adding equations (6) and (11),

$$2A^2 = \sqrt{(R^2 + L^2\omega^2)(G^2 + C^2\omega^2)} + (RG - LC\omega^2) \tag{12}$$

or

$$A = \sqrt{\frac{1}{2}\sqrt{(R^2 + L^2\omega^2)(G^2 + C^2\omega^2)} + \frac{RG - LC\omega^2}{2}} \tag{13}$$

Subtracting equation (6) from equation (11),

$$2B^2 = \sqrt{(R^2 + L^2\omega^2)(G^2 + C^2\omega^2)} - (RG - LC\omega^2) \tag{14}$$

or

$$B = \sqrt{\frac{1}{2}\sqrt{(R^2 + L^2\omega^2)(G^2 + C^2\omega^2)} - \frac{RG - LC\omega^2}{2}} \tag{15}$$

Since B is the phase or wave length constant, or is the retardation angle (in circular radians), the velocity of the propagation of a wave of angular velocity, ω, is in general

$$V = \frac{\omega}{B} \tag{16}$$

or the time retardation, t (in seconds), per unit length is

$$t = \frac{B}{\omega} \tag{17}$$

The wave length, λ, is

$$\lambda = \frac{2\pi}{B} \tag{18}$$

12.21 Loaded Circuit—No Leakance.—Let us now consider the special case in which $G = 0$ and the inductive reactance $L\omega$ is large as compared with the series resistance R. This case corresponds closely to that of a well-insulated open wire line.

When G is zero, (13) reduces to

$$A = \sqrt{\frac{C\omega}{2}\sqrt{R^2 + L^2\omega^2} - \frac{LC\omega^2}{2}} \tag{19}$$

If, furthermore, $L^2\omega^2$ is large as compared with R^2, it is known from the binomial theorem that

$$\sqrt{L^2\omega^2 + R^2} \doteq L\omega + \frac{R^2}{2L\omega} \tag{20}$$

where the sign ($\doteq$) denotes approximate equality, the two quantities becoming equal to each other when the ratio of $L^2\omega^2$ to R^2 becomes infinitely large. (See formula (60), Appendix A.)

Putting the value of (20) in (19),

$$A \doteq \frac{R}{2}\sqrt{\frac{C}{L}} \tag{21}$$

This equation is interesting in that it shows that in a circuit having negligible leakance G and large inductive reactance $L\omega$—such as a smoothly loaded line—the attenuation constant A is directly proportional to the resistance R. It is also proportional to the square root of the capacity C of the circuit and inversely proportional to the square root of the inductance L of the circuit. Although it follows from the above that, in general, adding inductance to a circuit—which is called *loading a circuit*—lowers the attenuation constant and so improves its efficiency, there is, in practice, a pretty definite limit to the amount of inductance which should be added due (1) to the extra cost of loading such a circuit, (2) to the presence of leakance G, and (3) to the fact that it is impossible to increase the inductance without at the same time increasing the total resistance R of the circuit.

12.22 Loaded Circuit—with a Small Amount of Leakance.—There is another special case which may be derived from formula (13) and which is frequently of interest. This case corresponds to that of a loaded circuit in which there is a small amount of leakance. In other words, it is assumed that $L^2\omega^2$ is large as compared with R^2 and that $C^2\omega^2$ is large as compared with G^2. This is then a case that approximates, more closely than the preceding one, the conditions existing in the ordinary loaded circuit. Since by the binomial theorem—equation (1) of Appendix A—

$$(a+b)^{1/2} = a^{1/2} + \frac{b}{2a^{1/2}} - \frac{b^2}{8a^{3/2}} + \cdots \tag{22}$$

expanding (13) gives

$$A \doteq \sqrt{\frac{1}{2}\left[L\omega + \frac{R^2}{2L\omega} - \frac{R^4}{8L^3\omega^3}\right]\left[C\omega + \frac{G^2}{2C\omega} - \frac{G^4}{8C^3\omega^3}\right] + \frac{RG - LC\omega^2}{2}} \tag{23}$$

or

$$A \doteq \sqrt{\frac{1}{2}\left[\frac{R^2C}{2L} + RG + \frac{G^2L}{2C} + \frac{R^2G^2}{4LC\omega^2} - \frac{LG^4}{8C^3\omega^2} - \frac{R^4C}{8L^3\omega^2} + \frac{R^2G^2}{16LC\omega^4}\left(\frac{R^2G^2}{4L^2C^2\omega^2} - \frac{R^2}{L^2} - \frac{G^2}{C^2}\right)\right]} \tag{24}$$

Neglecting all terms after the third, (24) reduces to

$$A \doteq \left(\frac{R}{2} + \frac{LG}{2C}\right)\sqrt{\frac{C}{L}} \tag{25}$$

This formula is very useful in enabling one to get very quickly the approximate attenuation constant of a loaded circuit. This formula reduces—as it of course should—to formula (21) when G is made equal to zero.

12.23 Non-Loaded (Small Gauge) Cable.—In a similar way, an approximate expression is derived for the attenuation constant of a non-loaded circuit. In this derivation it is assumed that R^2 is large compared with $L^2\omega^2$ and that $C^2\omega^2$ is large compared with G^2. With these assumptions, equation (13) and the binomial theorem give:

$$A \doteqdot \sqrt{\frac{1}{2}\left[R + \frac{L^2\omega^2}{2R} - \frac{L^4\omega^4}{8R^3}\right]\left[C\omega + \frac{G^2}{2C\omega} - \frac{G^4}{8C^3\omega^3}\right] + \frac{RG - LC\omega^2}{2}} \tag{26}$$

or

$$\begin{aligned} A \doteqdot \sqrt{\frac{1}{2}\Big[RC\omega + \frac{RG^2}{2C\omega} - \frac{RG^4}{8C^3\omega^3} + RG + \frac{L^2C\omega^3}{2R} + \frac{L^2G^2\omega}{4CR}} \\ \overline{- \frac{L^4C\omega^5}{8R^3} - LC\omega^2 + \frac{L^4G^4\omega}{64R^3C^3} - \frac{L^2G^4}{16RC^3\omega} - \frac{G^2L^4\omega^3}{16R^3C}\Big]} \end{aligned} \tag{27}$$

Whence, neglecting all terms after the second

$$A \doteqdot \sqrt{\frac{RC\omega}{2}\left[1 + \frac{1}{2}\left(\frac{G}{C\omega}\right)^2\right]} \tag{28}$$

It is seen from the above expression that the attenuation constant in this case increases approximately in proportion to the square root of the resistance whereas in the case of a loaded circuit (see formula (21)) it increases directly as the resistance.

If the leakance G is very small—as it is in many types of non-loaded cables—the above expression reduces to

$$A \doteqdot \sqrt{\frac{RC\omega}{2}} \tag{29}$$

This formula holds very closely for small gauge non-loaded paper-insulated cables.

12.24 Distortionless Transmission.—If, in the general expressions for A and B (see equations (13) and (15)), the quantity $2RGLC\omega^2$ is added to and subtracted from the product $(R^2 + L^2\omega^2)(G^2 + C^2\omega^2)$, the expressions for A and B can be thrown into the form

$$A = \sqrt{\tfrac{1}{2}\left[\sqrt{(RG + LC\omega^2)^2 + (LG - RC)^2\omega^2} + (RG - LC\omega^2)\right]} \tag{30}$$

and

$$B = \sqrt{\tfrac{1}{2}\left[\sqrt{(RG + LC\omega^2)^2 + (LG - RC)^2\omega^2} - (RG - LC\omega^2)\right]} \tag{31}$$

If the primary constants have such values that

$$LG - RC = 0 \quad \text{or} \quad \frac{L}{R} = \frac{C}{G} \tag{32}$$

then equation (30) becomes

$$A = \sqrt{RG} \tag{33}$$

and equation (31) reduces to

$$B = \omega\sqrt{LC} \tag{34}$$

When the conditions given in (32) have been fulfilled, the circuit is said to be a *distortionless circuit* inasmuch as the attenuation constant (see equation (33)) is then independent of the frequency. This condition was first pointed out and discussed by Heaviside.* It is apparent that in the distortionless case the velocity of propagation of waves of all frequencies is the same and is

$$V = \frac{\omega}{B} = \frac{\omega}{\omega\sqrt{LC}} = \frac{1}{\sqrt{LC}} \tag{35}$$

This is essentially the velocity of light in the case of a non-loaded circuit. At 800 cycles, the velocity of wave propagation on different types of circuits, as computed by formula (16) and using the circuit constants commercially encountered, is approximately as follows:

Type of circuit	Velocity-miles per second
N.L. No. 12 N.B.S.G.	174,000
N.L. No. 8 B.W.G.	178,000
N.L. No. 19 A.W.G.	39,000
N.L. No. 22 A.W.G.	28,000
L. No. 12 N.B.S.G.	56,000
L. No. 8 B.W.G.	53,000
X.L.L.	20,000
M.H.L.	10,000

In the above table N.L. designates *non-loaded*, M.H.L. *medium heavy loaded* and X.L.L. *extra light loaded*. The very low speed of wave propagation on loaded circuits is often an objection to their use, especially on long repeatered circuits where *echo effects* and transients may become serious.†

12.3 Simplified Expressions for the Iterative Impedance of a Line Having Smoothly Distributed Constants.—It is seen from (3) that the iterative impedance of a circuit is not, in general, independent of the frequency. For example, in a well-insulated non-loaded cable circuit the leakance G and inductance L are both so small that they can be neglected and the expression for the iterative impedance can be written

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* Heaviside's Electrical Papers, Vol. II, p. 123.

† For a more detailed discussion of this phase of the subject, as well as for a general view of some of the problems of telephone transmission over long circuits, reference may be made to an article on "Telephone Transmission over Long Cable Circuits," by A. B. Clark, *Journal of the A. I. E. E.*, Vol. XLII, January, 1923, and to an article by H. S. Osborne entitled "Telephone Transmission Over Long Distances," *Journal of the A. I. E. E.*, Vol. XLII, October, 1923.

$$Z_K \doteq \sqrt{\frac{R}{jC\omega}} \tag{36}$$

In this case the impedance decreases with increasing frequency. If, however, the inductive reactance $L\omega$ is large and the leakance G is small, the iterative impedance is seen from (3) to approach the value

$$Z_K \doteq \sqrt{\frac{L}{C}} \tag{37}$$

that is, the iterative impedance becomes practically independent of the frequency. It is, moreover, apparent from formula (3) that an increase in the inductance of a circuit results in an increase in the magnitude of the iterative impedance and a decrease in its phase angle. For example, the 800-cycle iterative impedance of a non-loaded No. 19 gauge cable circuit is approximately $500 \,/\overline{45^\circ}$, and that of a non-loaded open wire line is about $700 \,/\overline{14^\circ}$ ohms. The corresponding value for the iterative impedance of such circuits, if loaded, might be as high as 2,000 ohms and the phase angle might be only one or two degrees negative, i.e., $Z_K \doteq 2{,}000 \,/\overline{1^\circ.5}$ ohms. Hence, the iterative impedance of a loaded line at voice frequencies is very close, in absolute magnitude, to that which the line would have if the frequency were infinitely great, and its value is given very closely by $\sqrt{L/C}$.

12.4 Equivalent *T* Network of a Line Having Smoothly Distributed Constants.—It can be shown from equations (3) and (4) that

$$PZ_K = R + jL\omega \tag{38}$$

and

$$\frac{Z_K}{P} = \frac{1}{G + jC\omega} \tag{39}$$

Hence

$$Z_K \tanh\frac{P}{2} = Z_K \frac{P}{2}\,\frac{\tanh\frac{P}{2}}{\frac{P}{2}} = \frac{R + jL\omega}{2} \times \frac{\tanh\frac{P}{2}}{\frac{P}{2}} \tag{40}$$

and

$$\frac{Z_K}{\sinh P} = \frac{Z_K}{P} \times \frac{P}{\sinh P} = \frac{1}{G + jC\omega} \times \frac{P}{\sinh P} \tag{41}$$

Consequently, the equivalent T network of a line having smoothly distributed constants may be represented as shown in Fig. 2. Furthermore, since for small values of P, $\frac{\tanh P/2}{P/2}$ and $\frac{P}{\sinh P}$ will, in general, each approach unity, lines of relatively small electrical lengths can be closely represented by a T network in which one-half of the total resistance and inductance are placed in each of the series arms and the total leakance and capacity is concentrated in the shunt arm. Artificial lines and cables are frequently so constructed, as indicated in Fig. 3.

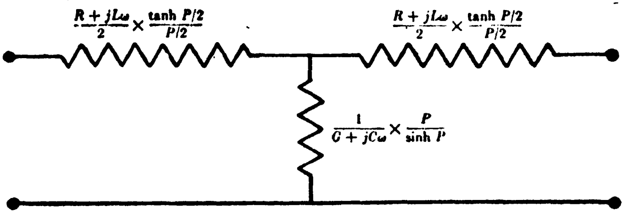
$\frac{R + jL\omega}{2} \times \frac{\tanh P/2}{P/2}$
$\frac{R + jL\omega}{2} \times \frac{\tanh P/2}{P/2}$
$\frac{1}{G + jC\omega} \times \frac{P}{\sinh P}$

FIG. 2. Equivalent T network of a line having smoothly distributed constants.

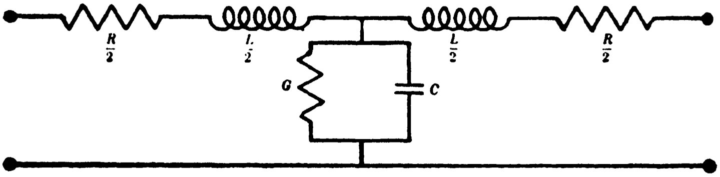

FIG. 3. T network approximately representing a short line having smoothly distributed constants, R, L, G and C.

If the absolute magnitude of the propagation constant of a line is more than .2 or .3, the factors $\frac{\tanh P/2}{P/2}$ and $\frac{P}{\sinh P}$ begin to depart appreciably from unity, and it is then usually advisable to represent only a certain fraction of the total length of line by one T section—the entire line being, in such a case, simulated by a series of such T networks.*

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* For a fuller discussion of artificial lines see "Artificial Electric Lines, Their Theory, Mode of Construction and Uses," by A. E. Kennelly.

CHAPTER XIII

LOADED CIRCUITS

13.0 It can be easily shown by means of the formulae given in Sect. 12.2 * that up to a certain point the attenuation constant of a circuit is decreased, and hence the transmission efficiency of the circuit is increased, if series distributed inductance can be uniformly introduced into the circuit. This method of increasing the efficiency of a circuit was first suggested by Heaviside in 1887 † who did not, however, point out any practical way of introducing the inductance.

13.1 Continuous Loading.—One method ‡ of adding inductance to a transmission circuit is by winding wire or tape of iron, or other magnetic material, around the conductor to be loaded. Rigorous formulae holding for such a circuit have already been derived in Sect. 12.1. The objections to continuous loading by such methods are (1) its excessive cost, (2) the relatively small amount of loading (ordinarily less than .03 henry per mile) that can be introduced into the circuit, (3) the relatively large increase in effective resistance accompanying such loading, and (4) the difficulty of predicting the constants and hence the efficiency of such a circuit—small differences in mechanical treatment or pressure between the tape and the conductor making large differences in the a-c. constants of the circuit. This last objection is particularly serious on circuits containing repeaters where a definite and smooth line impedance-frequency characteristic is necessary for proper balance conditions.

Continuous loading is, however, frequently used on submarine cables—as for example in the Key West-Havana submarine cable§—for the reason that in deep water the difficulty of making water-tight joints at the loading points makes coil loading undesirable. Moreover, repairs on a deep sea cable with lumped loading introduce irregularities which are practically unavoidable and which may be so large that for such cables continuous loading is sometimes even preferred for uniformity.

When a circuit is loaded by such a method, its inductive reactance $L\omega$ is usually large as compared with its resistance R and the susceptance

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* See also Chapter IX of J. G. Hill's "Telephonic Transmission."

† See *The Electrician*, Vol. XIX, p. 79, 1887.

‡ First suggested by C. E. Krarup, *Electrotechnische Zeitschrift*, p. 344, 1902.

§ For a description of this cable, see paper by Martin, Anderegg and Kendall, on "The Key West-Havana Submarine Telephone Cable System," *Journal A. I. E. E.*, Vol. 41 (1922), pp. 184–202.

$C\omega$ is large as compared with the leakance G. Under these conditions the formula for the attenuation constant of the circuit may be very closely approximated by equation (25) of Sect. 12.22.

13.2 Lumped Loading.—Pupin and Campbell both proved, at approximately the same time,* that the insertion of inductance coils at uniform intervals in a telephone circuit would, under certain conditions, affect the attenuation constant of a circuit in essentially the same way as if the inductance were uniformly distributed throughout the circuit. The conditions for this equivalence, which were fully developed in Campbell's paper in the *Philosophical Magazine,* are that the frequency to be transmitted shall not exceed a certain frequency which is called the *critical* or *cut-off frequency.* In order to get a concrete idea of these conditions, a simple derivation will be given for the formulae for the propagation constant and the iterative impedance of such a loaded circuit.

Let N be the number of miles between the loading coils, P the propagation constant per mile and Z_k the iterative impedance of the non-loaded circuit. Also let Z_c be the impedance of each loading coil. One section of the loaded line—terminated at mid-coil—may then be represented as shown in Fig. 1. An infinite number of such structures would evidently represent an infinite length of the loaded circuit. The propagation constant of the N miles of the non-loaded line may be obtained from the second expression of equation (7) of Chapter XI and is given by

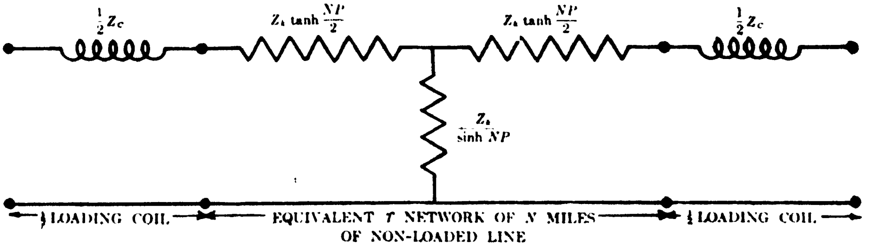

FIG. 1. Structure representing one section of a loaded circuit terminated in a mid-coil.

$$\cosh(NP) = 1 + \frac{Z_1}{2Z_2} \qquad (1$$

where

$$\frac{1}{2}Z_1 = Z_k \tanh\frac{NP}{2} \qquad (2$$

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* See papers by M. I. Pupin in the *Proceedings of the A. I. E. E.*, Vol. XVII, May 19, 1900, p. 445; and by G. A. Campbell in the *Philosophical Magazine* for March, 1903.

and

$$Z_2 = \frac{Z_k}{\sinh (NP)} \tag{3}$$

Z_1 and Z_2 being, respectively, the series and shunt impedances of the equivalent ladder structure. Similarly, for the loaded line the relation is

$$\cosh (NP') = 1 + \frac{Z_1'}{2Z_2} \tag{4}$$

in which

$$\frac{1}{2}Z_1' = Z_k \tanh \frac{NP}{2} + \frac{1}{2}Z_c = \frac{1}{2}Z_1 + \frac{1}{2}Z_c \tag{5}$$

Whence

$$\cosh (NP') = 1 + \frac{Z_1 + Z_c}{2Z_2} = \left(1 + \frac{Z_1}{2Z_2}\right) + \frac{Z_c}{2Z_2} = \cosh (NP) + \frac{Z_c}{2Z_2} \tag{6}$$

or

$$\begin{aligned} P' &= \frac{1}{N}\cosh^{-1}\left[\cosh (NP) + \frac{Z_c}{2Z_2}\right] \\ &= \frac{1}{N}\cosh^{-1}\left[\cosh (NP) + \frac{Z_c \sinh (NP)}{2Z_k}\right] \end{aligned} \tag{7}$$

which is the so-called *Campbell formula* for a line loaded with series coils, and gives rigorously the propagation constant P' per mile of loaded line in terms of the propagation constant P per mile of the non-loaded line, the iterative impedance Z_k of the non-loaded line, the number of miles N between the loading coils and the impedance Z_c of each loading coil.

By means of transformations similar to those shown in Appendix C, it can be shown from formula (7) that

$$\begin{aligned} P' &= \frac{2}{N}\sinh^{-1}\left[\sinh \frac{NP}{2}\sqrt{1 + \frac{Z_c}{2Z_k \tanh \frac{NP}{2}}}\right] \\ &= \frac{2}{N}\tanh^{-1}\sqrt{\frac{2Z_k \tanh \frac{NP}{2} + Z_c}{2Z_k \coth \frac{NP}{2} + Z_c}} \\ &= \frac{2}{N}\cosh^{-1}\left[\cosh \frac{NP}{2}\sqrt{1 + \frac{Z_c \tanh \frac{NP}{2}}{2Z_k}}\right] \end{aligned} \tag{8}$$

It is, however, seldom necessary or desirable to calculate the propagation constant from any of the above formulae. The following procedure, when a rigorous determination of the propagation constant is necessary is generally used. The value of one-half of the series arm ($\frac{1}{2}Z_1'$) of the equivalent T network is determined by adding to one-half the coil impedance ($\frac{1}{2}Z_c$) the value of Z_k tanh $NP/2$—which in many practical cases can be regarded as one-half the total resistance and inductive reactance, $N[(R + jL\omega)/2]$, of the N miles of circuit (see Sect. 12.4). The value of the shunt impedance Z_2 of the equivalent T network is then determined from the value of $Z_k/\sinh NP$ which likewise—as pointed out in Sect. 12.4—can, in most all practical cases, be regarded as the impedance, $(1/N)[1/(G + jC\omega)]$, corresponding to the total shunt leakance and capacitance of the N miles of circuit. Then, knowing the values of Z_1' and Z_2 of the equivalent ladder type of structure, the attenuation constant—or the real part of the propagation constant—as well as the phase constant can be obtained from curves of the type shown in Figs. 11 and 12 of Appendix E. This method admits of as great rigor as may be desired and, in general, will be found to be much quicker and fully as accurate as the method of using the formulae given above in conjunction with charts or tables of complex quantities.

The mid-coil iterative impedance, Z_K, is—from equation (1) of Chapter XI

$$Z_K = \sqrt{\frac{Z_1'}{2}\left(\frac{Z_1'}{2} + 2Z_2\right)} = \sqrt{\frac{Z_1'}{2}\left(\frac{Z_1}{2} + \frac{1}{2}Z_c + 2Z_2\right)} \tag{9}$$

in which

$$\frac{Z_1'}{2} \equiv Z_k \tanh\frac{NP}{2} + \frac{1}{2}Z_c$$

From equations (1) and (7) of Chapter XI,

$$Z_K \coth\frac{NP}{2} = \sqrt{\frac{Z_1}{4}(Z_1 + 4Z_2)} \times \sqrt{\frac{Z_1 + 4Z_2}{Z_1}} = \frac{Z_1}{2} + 2Z_2$$

Hence

$$Z_K = \sqrt{\left(Z_k \tanh\frac{NP}{2} + \frac{1}{2}Z_c\right)\left(Z_k \coth\frac{NP}{2} + \frac{1}{2}Z_c\right)} \tag{10}$$

which gives, rigorously, the mid-coil iterative impedance.

In a similar way, it can be shown that the mid-section iterative impedance, Z_K', i.e., the impedance as measured mid-way between two loading coils, is

$$Z_K' = Z_K\sqrt{\frac{2Z_k + Z_c \coth\frac{NP}{2}}{2Z_k + Z_c \tanh\frac{NP}{2}}} = \left(Z_k \coth\frac{NP}{2}\right)\left(\tanh\frac{NP'}{2}\right) \tag{11}$$

It has been the practice* to load open wire circuits by inserting, at approximately eight mile intervals, loading coils having inductances of approximately 1/4 henry. This reduces the attenuation constant of such circuits to approximately one-half of that existing before the circuit was loaded. In other words, it should be possible to talk to approximately twice the distance over a loaded open wire circuit as over the same gauge of non-loaded circuit.

It was formerly the custom to design all loaded circuits such that their cut-off frequency was around 2,000 cycles—the cut-off frequency being a function of both the spacing and the inductance of the loading coils (viz., $f_c \doteq 1/\pi\sqrt{LC}$, where L is the inductance of the coils and C is the total shunt capacity in each loading section). This is still the approximate cut-off frequency of some open wire loaded circuits. The tendency recently, however, has been to raise this cut-off frequency inasmuch as it is now known from articulation tests (see Sect. 2.2) that there are important frequencies in the voice which lie appreciably above 2,000 cycles. On cable circuits it is now customary to space all loading coils 6,000 feet apart and to use coils either of .175 henry (in which case the loading is called *medium heavy loading* and gives a cut-off frequency of about 2,800 cycles and an iterative impedance of approximately 1,600 ohms) or else use coils of an inductance of .044 henry (in which case the loading is called *extra light loading* and gives a cut-off frequency of about 5,600 cycles and an iterative impedance of approximately 800 ohms). This is on the assumption that the capacity per loading section (6,000 ft.) is approximately .074 microfarad.

13.21 Loading Coils.—*Loading coils* are ordinarily wound on cores in the shape of annular rings or toroids. Cores are usually made up of a very large number of turns of fine iron wire, or of fine iron dust mixed with a binder and pressed into ring-shaped form.† Coils which are used to load the side circuits of a phantom circuit (see Sect. 19.1) have two balanced windings, each of which is placed in series with a wire of the circuit. These windings are connected in such a way that currents flowing down one line wire and back over the other will tend to build up flux in the core of the coil. Coils which are used for phantom loading have four balanced windings, each of which is put in one of the four wires of the phantom circuit. These windings are so connected that currents flowing in the side circuits will not build up any flux in the

* See article on "The Commercial Loading of Telephone Circuits in the Bell System," by B. Gherardi, *Trans. A. I. E. E.*, Vol. XXX, Part III (1911), pp. 1743–1773.

† See a paper by J. B. Speed and G. W. Elmen on "The Magnetic Properties of Compressed Powdered Iron," *Journal A. I. E. E.*, Vol. 40 (1921), pp. 596–609.

core of the coil. On the other hand, currents flowing through the phantom circuit will aid each other in building up flux in the core of the phantom coil. Consequently, the side circuit coil *loads* or offers inductance to the side circuit and simply adds a small d-c. resistance to the phantom circuit. Similarly, the phantom circuit coil adds inductance to the phantom circuit but offers only d-c. resistance to the side circuits. Other things being equal, a loading coil which has at telephonic frequencies a higher time constant (or L/R) than another loading coil is the more efficient of the two.

Loading coils are encased in iron pots and are mounted in manholes, if the circuit to be loaded is a cable circuit; or are mounted on the poles at the cross-arms, if the circuit to be loaded is an open-wire line.*

13.3 Artificial Networks Representing Loaded Circuits.—It is often desirable to have available for use in the laboratory an artificial line or structure which will have—to a sufficiently close degree—the same electrical characteristics as those of an actual line. Before considering the design of such a structure, to represent a loaded circuit, it will first be desirable to consider briefly the make-up of networks designed to simulate the characteristics of non-loaded circuits.

Structures which are used to represent non-loaded artificial cables or open wire lines are ordinarily made up by simply grouping one-half the total resistance and one-half the total inductance in each of the series arms of a T network and the total shunt capacitance in the shunt arm. The errors involved by the use of such a structure may be rigorously calculated by the formulae developed in Sect. 12.4. The errors involved by the use of such artificial structures increase with the geographical length of the circuit represented by the artificial structure and by the frequency considered; in other words, the errors increase with the *electrical length* of the circuit represented by such a structure.

For example, in most artificial cables used in the Bell System laboratories a two-mile section is the maximum employed to represent standard No. 19 gauge cable; i.e., a three-mile length of cable would be obtained by combining two T networks, the one representing a two-mile section and the other a one-mile section. For No. 12 N.B.S.G. open wire line, however, a twenty-mile length is the greatest length which a single T section is permitted to represent in those cases where the ordinary voice-range of frequencies is employed.

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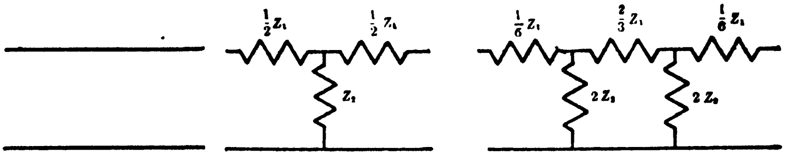
* For a more extended description of loading coils and their use, see an article on "Loading Coils," by W. Fondiller, *Western Electric News*, Aug., 1921, pp. 16–18.

The errors involved by the use of a single *T* network—with half the resistance and inductance in each series arm and the total capacity in the shunt arm—to represent two miles of standard cable (R = 88 ohms, C = .054 m.f. per mile) at 2,000 cycles are shown below.



	Actual cable	2-mile section	4-mile section
Prop. constant per mile	$.1728 + j\,.1728$	$.1746 + j\,.1710$	$.1750 + j\,.1704$
Iterative impedance	$254.6 - j\,254.6$	$262.4 - j\,247.2$	$250.4 - j\,256.8$

By a comparison of the values in columns 1 and 2, it is seen that the attenuation and phase constants of such an artificial cable (made up of a single *T* network and representing two miles of actual cable) deviate from those of the actual cable at 2,000 cycles by about 1 per cent. This deviation would of course be much less at the more important telephonic frequencies such as 800 or 1,500 cycles. It is also to be noted that the absolute magnitude of the iterative impedance of the artificial network is essentially the same as that of the actual cable which it is designed to replace, although the phase angle of the former deviates perceptibly from that of $/45^\circ$. In other words, the effective resistance component of the iterative impedance is about 3 per cent greater than and the effective reactance component is about 3 per cent less than that of the corresponding values for the actual cable circuit.

In the above table there is also given, in column 3, the propagation constant per mile and the iterative impedance of a network representing four miles of cable but made up as shown in Fig. 2B. This structure has its series arms made up of elements whose values are 1/6, 2/3 (or 4/6), and 1/6, respectively, of the total series impedance (R—in the case shown).

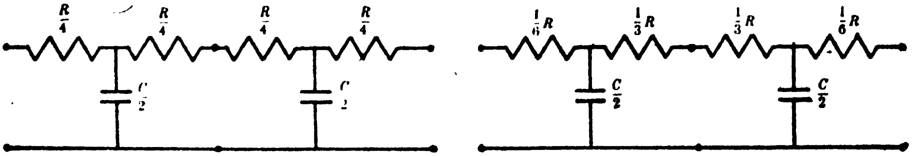


FIG. 2A FIG. 2B

FIGS. 2A and 2B. Fig. 2A—Two symmetrical *T* networks approximately representing a uniform line of resistance *R* and capacity *C*. Fig. 2B—Two dissymmetrical *T* networks approximately representing a uniform line of resistance *R* and capacity *C*.

From the figures given in columns 1 and 3 in the table, it is seen that the iterative impedance of the structure shown in Fig. 2B is much closer to that of the actual cable than is that of the structure shown in Fig. 2A—the data for which is given in column 2 of the table.

The structure in Fig. 2B has been mentioned here not so much on account of its application to the case where it is to represent a non-loaded circuit—for its propagation constant does not simulate that of the actual cable particularly closely—but on account of its application in those cases where a close representation of a *loaded* circuit must be made up in the laboratory. The use of such a network in the construction of an artificial loaded circuit was first suggested by Campbell and its advantages for this purpose over that of the ordinary symmetrical *T* network may be seen by referring to Figs. 3A, 3B and 3C as well as to the data given below.

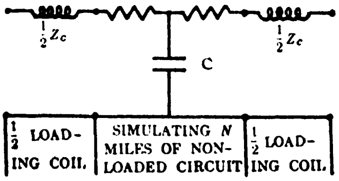


Fig. 3A

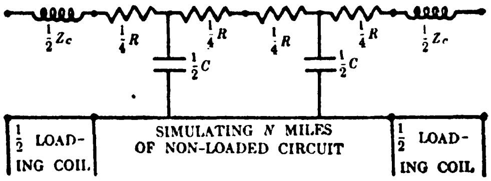


Fig. 3B

FIG. 3A. Single *T* network simulating *N* miles of non-loaded circuit terminating at each end in ½ loading coil.

FIG. 3B. Two symmetrical *T* networks simulating *N* miles of loaded circuit terminating at each end in ½ loading coil.

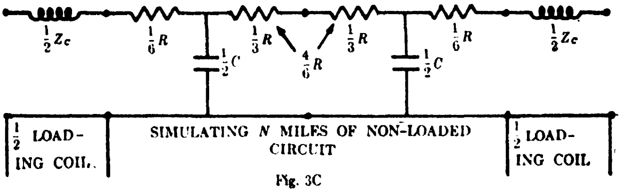


Fig. 3C

FIG. 3C. Two dissymmetrical *T* networks simulating *N* miles of non-loaded circuit terminating at each end in ½ loading coil.

In Fig. 3A, the non-loaded circuit between loading coils is represented by a single *T* network and in Fig. 3B by two symmetrical *T* networks while in Fig. 3C the non-loaded circuit between the loading coils is represented by two dissymmetrical *T* networks, the series arms of each of which are respectively ⅙*R* and ⅓*R*. In the following table are given the propagation constants and the iterative impedances of the various structures shown above when each is designed to represent a No. 16 gauge loaded cable, the cable proper having a distributed resistance of 42.12 ohms per mile, .062 mf. per mile, and

.001 henry per mile. The loading coils are assumed to introduce, at 2,000 cycles, an impedance of $10 + j\,.175\,\omega$ ohms and to be spaced every 1.66 miles. With these assumptions the following data are obtained.

	Prop. constant per mile	Iterative impedance
1 *T* section	$.0340 + j\,1.208$	$698\,\underline{/1^\circ 35'}$
2 symmetrical *T* sections	$.0236 + j\,1.208$	$697\,\underline{/52'}$
2 dissymmetrical sections, Campbell arrangement	$.0200 + j\,1.208$	$697\,\underline{/5'}$
Actual loaded cable	$.0201 + j\,1.214$	*

From the foregoing table it is seen that if an artificial cable or line is to be made up to represent a series lumped loaded circuit, the networks representing the non-loaded portion of the actual circuit—between loading coils—should be made up as indicated in Fig. 3C, viz., in two dissymmetrical *T* networks, the series arms of each of which are respectively ⅙ and ⅓ of the total series impedance (resistance, and distributed inductance—if the latter is not negligible) and the shunt arms are each twice the total shunt impedance of the non-loaded circuit.

* The iterative impedance is not given here as it is evidently a function of the position in the line (i.e., distance from a loading coil) at which it is measured. It might be remarked, however, that the iterative impedance of such a cable circuit will, in general, have a slightly negative phase angle and hence will be fully as closely simulated by the Campbell arrangement as it will be by either of the other artificial structures referred to in the table.

CHAPTER XIV

TELEPHONE REPEATERS

14.0 A telephone repeater is a device for amplifying voice currents. In its usual form it consists essentially of a repeater element for amplifying the voice currents and a repeater circuit which automatically enables this amplification to be accomplished in both directions over a telephone circuit.

Its use permits telephonic communication over distances for which the costs of efficient line conductors would otherwise be prohibitive, and also admits of more economical circuits for more moderate distances.

14.1 Description of Elements.*—All forms of repeater elements which have so far been adapted to commercial requirements are of the *unilateral mutual impedance* type. In this type a current flowing in the input circuit produces a potential in the output circuit, but a current flowing in the output circuit induces no potential in the input circuit.

The requirements for an ideal repeater element are as follows:

1. The ratio of output to input power must be independent of both the frequency and the magnitude of the power.
2. The input and output impedances must also be independent of the magnitude or frequency of the impressed power.
3. The element must give a large amplification. It is ordinarily desirable to have a power amplification, for speech-frequency currents, of 100 times or more, although for certain uses elements giving smaller amplifications might be employed.
4. The local sources of energy must not involve potentials unsafe for use in the telephone plant.
5. The element must be insensitive to external electrical disturbances and should not produce external fields which will cause crosstalk with adjacent circuits.
6. The element should be of long life, and so constant and reliable in service as not to demand exceptional maintenance service.
7. In size, first cost, cost of power and of maintenance, the element must conform to the economic conditions of an established plant.

14.11 Mechanical Repeater Element.—The first repeater element to go into commercial service was the mechanical type, consisting essentially of a telephone receiver coupled to a sensitive microphone supplied with direct current in the usual manner. In this manner weak incoming waves serve to control the release of relatively large waves of the same frequency and relative amplitudes.

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* A large part of the material in this section is taken from an article on "Telephone Repeaters," by B. Gherardi and F. B. Jewett, *Proc. A. I. E. E.*, 1919, pp. 1255–1313.

The chief bar to its more extensive use is its failure (due to initial friction and fixed losses) to respond to incoming waves below a critical value. The tendency of microphone buttons to *breathe*, due to heating and packing, and the distortion inherent in carbon buttons make even the best-designed of such devices difficult to maintain when several are operated in one system.

14.12 Vacuum Tube Type of Element.—The thermionic *vacuum tube** repeater element, or *audion*, as it is sometimes called, consists of an evacuated vessel containing three electrodes, from one of which, the filament, a thermionic emission of electrons is obtainable. The filament is heated by the passage of a current as shown in Fig. 1. The other two electrodes are a plate and a grid. If a battery is connected to the filament and plate, as shown, so as to make the latter positive with respect to the former, then a current will flow in the circuit so formed. The electrons emitted at the filament are drawn across the intervening vacuum by the electric field which the battery B in the plate circuit has established. If an electromotive force is now applied between the grid and the filament, as by the source marked E in the figure, the field between plate and filament is altered and the current in the plate circuit is correspondingly altered. If the grid is made positive with respect to the filament, more electrons are urged across the space between grid and filament. While some of these electrons strike the grid and thus result in a current in that circuit, by far the greater number continue through the meshes of the grid to the plate. The result is an increased current in the plate circuit. Conversely, if the grid is made negative, there results a decrease in the plate current. In this case, however, no current

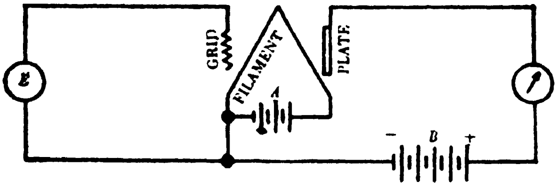

FIG. 1. Simplified circuit using a 3-element vacuum tube.

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* See the "Thermionic Vacuum Tube and its Applications," by H. J. Van Der Bijl. A simple and condensed discussion of the action of the vacuum tube is given in Chapter III of "Radio Communication," by J. Mills.

flows in the grid circuit because electrons can be drawn to an electrode only if it is positive with respect to the source of the electrons.

The characteristic relation between the grid voltage E and the plate current I_B is that shown in Fig. 2.

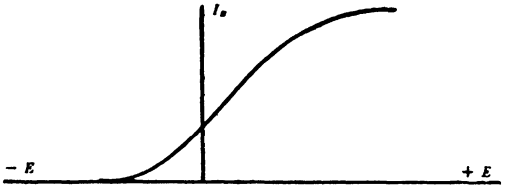

FIG. 2. Grid voltage-plate current characteristic.

If the plate voltage is altered, the form of the curve is not altered but the magnitude of the current is changed as illustrated in Fig. 3,

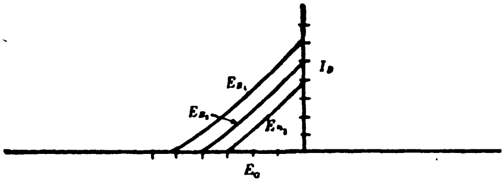

FIG. 3. Family of grid voltage-plate current characteristics.

which shows a family of such characteristics. It can be shown from such curves that the voltage which must be applied to the grid in order to reduce the plate current to zero is always the same fraction of the voltage applied in the plate circuit. Hence, it appears that a large change in the plate circuit current may be produced by a relatively small alteration of the voltage applied to the grid circuit. The device thus amplifies electrical impulses impressed on the grid circuit.

As long as the grid is kept negative no current can flow in that circuit, and any alterations in its voltage are unaccompanied by any current variation and hence are entirely wattless. Such variations are accompanied by current variations in the plate circuit and result in an energy expenditure in that circuit. The telephone efficiency is thus seen to be practically infinite, since an energy output may be obtained by a wattless variation of the input voltage. The practical limitations to a complete realization of this ideal efficiency lie in the design of voltage transformers and the possession by the tube of a finite leakage conductance and capacity reactance.

For use in amplifying telephone currents the voltage E, in Fig. 1, is replaced by an *input transformer* whose primary is connected to the source of speech. The galvanometer is replaced by one winding of an *output transformer*, the other winding of which is connected to a line or to the receiving device.

14.2 Description of Repeater Circuits.

14.21 One-Way Repeater Circuit.—If one telephone line is connected to the input of a repeater element of the unilateral mutual impedance type, and another line to its output, we have a repeater circuit capable of transmitting and amplifying speech in one direction only, or a one-way repeater circuit.

14.22 "21" Repeater Circuits.—Unless two pairs of wires—or two channels—are run to every subscriber, it is necessary that repeating systems give amplification in both directions. All forms of such circuits in common use depend upon the Wheatstone bridge arrangement, which is also shared in principle by duplex telegraph and invariable anti-side-tone substation circuits. *If, for example, in any invariable anti-side-tone substation circuit the receiver is considered to be the input element of a repeater and the transmitter to be the output element of the repeater, there is produced a* 21 (*that is, two-way, one-repeater*) *repeater circuit provided the balancing network of the substation circuit is replaced by the second line in the repeater circuit.* Such a transformation of an anti-side-tone circuit into a 21 repeater circuit is shown in Figs. 4A and 4B. In Fig. 4B the receiver and transmitter correspond to the input and output of any type of repeater element. Such circuits have many interesting properties, one of which is most fundamental to two-way repeating; namely, if the lines west and east are equal in impedance—both in magnitude and in phase—the output from the transmitter produces no input into the receiver.

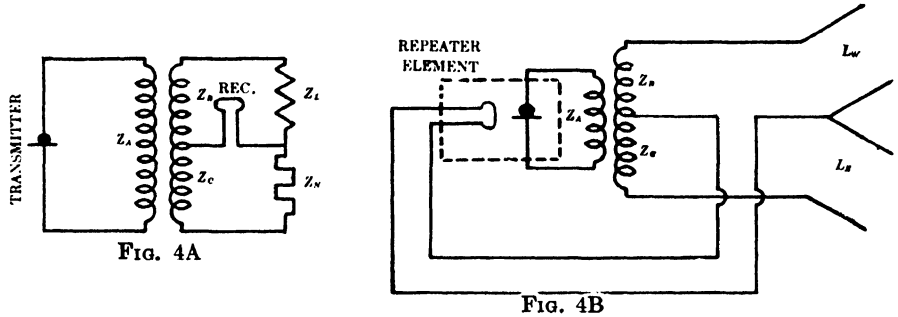

FIG. 4A

FIG. 4B

FIGS. 4A and 4B. Circuits showing the analogy between an anti-side-tone substation circuit and a 21 type repeater circuit.

It follows that part of the energy coming in on either line will go directly into the transmitting or output element and will be dissipated as heat. Approximately one-half of the incoming energy, however, will go into the receiving or input element which, by controlling the amount of energy given out by the output element, transmits amplified energy into both lines thereby producing two-way amplification. If the two lines are not equal in impedance, that is, if the repeater is "unbalanced," the outgoing energy produces a potential across the receiver or input element and a circulating current is thereby set up in the repeater circuit. This circulating current may, if large enough, produce distortion in the amplified current and, if still larger, may produce *singing* or oscillations which will prevent the repeater from functioning properly.

In order to get a quantitative idea of the relation existing between the number of miles gain, which any 21 type of repeater circuit can give, and the amount of unbalance which exists between any two lines to which the repeater may be connected, consider the circuit shown in Fig. 5.

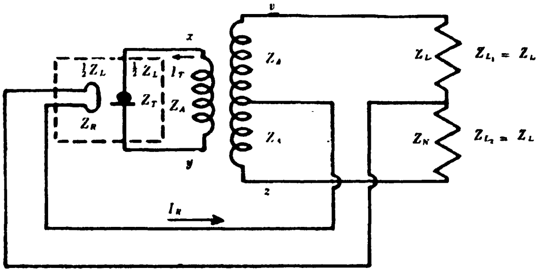

FIG. 5. Simplified diagram of a 21 type repeater circuit.

Referring to formula (82) of Appendix E, which applies to the corresponding type of anti-side-tone substation circuit, it is evident that if $Y = 1$ and if $Z_R = Z_T$ (whence $Z_A = Z_D$) then

$$Z_L : Z_T : Z_R = 2 : 1 : 1 \tag{1}$$

Assume an ideal transformer and an e.m.f. E acting in Z_T, it is then apparent from the equation (No. 73 of Appendix E) for I_R that

$$I_R = \frac{E(Z_L - Z_N)}{2Z_L(Z_L + Z_N)} \cdot \tag{2}$$

or

$$E_R \equiv I_R Z_R = \frac{E(Z_L - Z_N)}{2Z_L(Z_L + Z_N)} Z_R = \frac{E(Z_L - Z_N)}{4(Z_L + Z_N)} \tag{3}$$

If μ—sometimes called the *voltage amplification constant*—is the ratio of the voltage acting in the output of the repeater element Z_T to that absorbed in the input element Z_R, the return voltage E_T acting in Z_T is

$$E_T = \frac{E(Z_L - Z_N)}{4(Z_L + Z_N)} \mu. \tag{4}$$

If E_T is equal to or is greater in absolute magnitude than E, the circuit will evidently maintain a continuous current, in which case it is said to *sing* or *oscillate*. Equating E_T to E will then give the *singing point* or the limiting condition beyond which satisfactory operation of the repeater cannot be maintained. Equation (4) then becomes

$$\mu = \left|\frac{4(Z_L + Z_N)}{Z_L - Z_N}\right| \tag{5}$$

Let us now consider the *effective current amplification in the line, Z_N,* due to the insertion of such a repeater circuit. If no repeater were in the circuit and if the two lines were directly connected together, the current in Z_N, with an e.m.f. e in Z_L, would be

$$I_N = \frac{e}{Z_L + Z_N} \tag{6}$$

With the repeater in the circuit and an e.m.f. e in Z_L, the current in Z_R is $I_R = e/2Z_L$. The voltage across Z_R is

$$I_R Z_R = \frac{e}{2Z_L} \times \frac{Z_L}{2} = \frac{e}{4} \tag{7}$$

The amplified e.m.f. acting in Z_T is then $e\mu/4$. Whence

$$I_T = e\mu/4Z_L \tag{8}$$

Since the impedance, in each direction, at the terminals x–y is approximately $Z_L/2$, the voltage across these terminals is

$$(e\mu/4Z_L)(Z_L/2) \doteqdot (e\mu)/8 \tag{9}$$

Since, however, there are twice as many turns across the terminals v–z as there are between the terminals x–y, the voltage across v–z is $e\mu/4$ and the current in Z_N is

$$I'_N \doteqdot \frac{e\mu}{4(Z_L + Z_N)} \tag{10}$$

This amplified current is, however (see equation (6)), $\mu/4$ times as large as if no repeater circuit had been used. Hence, the effective current amplification due to the repeater is, from (5),

$$(\mu/4) \doteqdot |(Z_L + Z_N)/(Z_L - Z_N)| \tag{11}$$

Whence, the number of miles gain it is possible to get out of an ideal "21" type of repeater circuit, before it actually sings, is given by the relation

$$L \doteqdot 21.13 \log_{10} |(Z_L + Z_N)/(Z_L - Z_N)| \tag{12}$$

Curves plotted from this relation are given in Fig. 6.

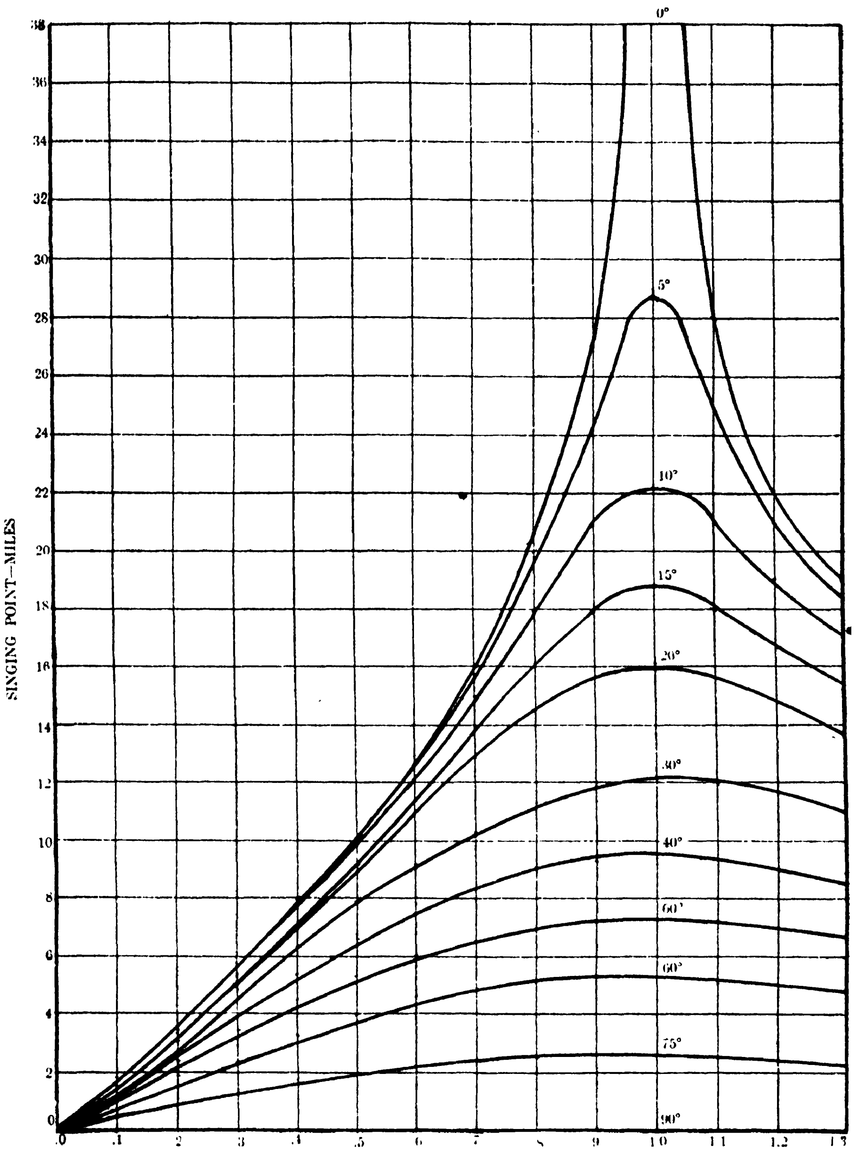

FIG. 6. Singing point versus the vector ratio of the line impedances in a 21 type repeater circuit.

An inspection of these curves shows the great importance of a close balance, in both magnitude and phase, of the two lines of a 21 repeater circuit. For example, if one line has an impedance that deviates in absolute magnitude from that of the other line by 10 per cent and if the phase angles of the two impedances differ by 10°, the maximum possible gain that can be obtained from the repeater circuit before singing takes place is 21 miles. This represents roughly the best condition which might be encountered in commercial practice and would ordinarily only be attained if a 21 repeater circuit were inserted at the middle of a long uniform circuit. Moreover, due to the fact that there is a considerable amount of distortion in a repeater circuit just before actual singing takes place, it is not feasible to operate such a repeater circuit at the full gain which could be permitted before singing actually takes place. As a matter of fact, with the balancing networks which can ordinarily be designed,* a gain of from 15 to 18 miles is about the maximum which may be expected to be obtained from a single 21 type repeater circuit.

14.23 "22" Repeater Circuits.—For many conditions a "22" repeater (i.e., two-way, two-repeater) circuit is much more desirable than a 21 repeater circuit. Such a circuit can be constructed from any two invariable anti-side-tone substation circuits provided one considers the receiver of the first anti-side-tone circuit and the transmitter of the second anti-side-tone circuit to be respectively the input and output elements of one amplifier and similarly, the receiver of the second anti-side-tone circuit and the transmitter of the first anti-side-tone circuit to be respectively the input and output elements of the second amplifier. Considering the form of anti-side-tone circuit referred to in Sect. 14.22, the corresponding 22 repeater circuit arrangement is shown in Fig. 7. Such a circuit will evidently not sing provided *either* $Z_L = Z_N$ *or* $Z_L' = Z_N'$. Hence, it is a better type of circuit from a singing standpoint than a 21 type of circuit.

The action of such a circuit is evidently as follows. If some one is assumed to be talking in the line east, L_E, an e.m.f. acting in Z_L will be generated and one-half of the incoming power will be dissipated in the receiver or input element Z_R' and the other half will be wasted in the impedance of the transmitter or output element Z_T. The power dissipated in Z_R' will be amplified by the repeater R_2 and the amplified output from Z_T' will produce equal voltages across the windings $C_3 - C_4$ and $D_3 - D_4$. Whence, if the impedance of line L_W or Z_L' is equal to that of the balancing network Z_N', there will be no potential across the terminals of Z_R and all of the amplified power will be dissipated in either Z_L' or Z_N'.

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* For methods of designing networks to simulate the impedance of smooth lines and of loaded circuits, see papers by R. S. Hoyt on the "Impedance of Smooth Lines and Design of Simulating Networks," *The Bell System Technical Journal*, Vol. II, No. 2 (April, 1923), and "Impedance of Loaded Lines, and Design of Simulating and Compensating Networks," *Bell System Technical Journal*, July, 1924.

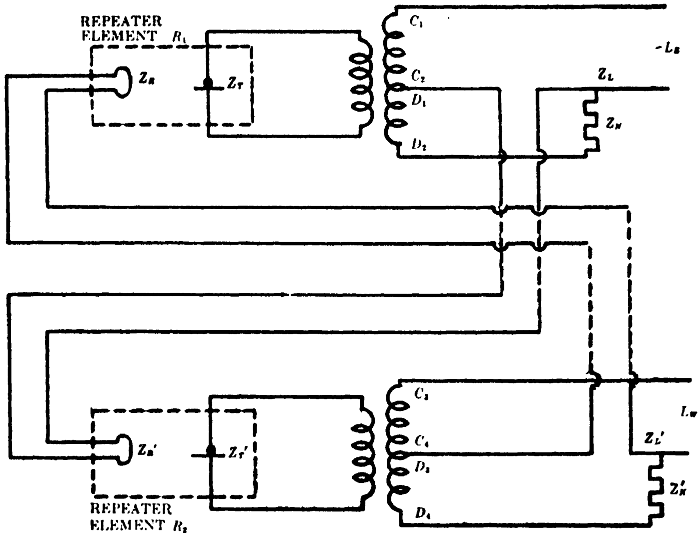

FIG. 7. Simplified diagram of a 22 type repeater circuit.

If, however, the line impedance Z_L' does not exactly equal Z_N', then there will be a slight difference of potential across the terminals of Z_R and the resulting amplified power from Z_T will produce equal potentials across the windings $C_1 - C_2$ and $D_1 - D_2$. Then if $Z_L = Z_N$, one-half of the amplified power from Z_T will be dissipated in the line impedance Z_L and the other half in the balancing network Z_N. If, however, the line impedance Z_L does not equal Z_N, then there will be a potential across the terminals of Z_R'. It is, therefore, evident that provided *either* $Z_L = Z_N$ *or* $Z_L' = Z_N'$ there can be no circulatory current around the loop including Z_R', Z_T', Z_R and Z_T. However, if Z_L' does not equal Z_N' and if Z_L does not equal Z_N, then there will be a certain amount of circulatory current. If the unbalances between the line impedances and the corresponding balancing networks are relatively great or if the loss in this loop circuit is less than the total amplification, there will be produced a self-sustaining current, or in other words the circuit will *sing*.

On the best high-grade lines, the balance of a 22 type repeater circuit can be maintained so as to permit an energy amplification of 100 times; this corresponds to a gain of 21 miles of standard cable, at 800 cycles. Assuming that the impedances of both lines deviate by the same amount from the impedances of their corresponding networks, the above gain would require that at all frequencies of efficient amplification the line impedances must not depart from those of their corresponding networks by more than approximately 10 per cent. As pointed out in the case of the 21 repeater circuit, the actual amplification must be appreciably less than that which will actually produce singing, or distortion of telephonic quality will result from a transient circulating current.

A typical 22 repeater circuit using vacuum tube repeater elements is shown in Fig. 8.

A comparison of the 21 and 22 type repeaters will establish the following points: 1. The 21 type repeater requires only half the power and less than half the apparatus and space needed by the 22 type repeater. 2. The 22 type repeater may be so equipped as to balancing networks that it will operate satisfactorily between *any* types of lines. The 21 type repeater requires additional apparatus in order to permit it to be used between lines of dissimilar character. 3. The 21 type repeater is not suited for tandem operation, as it feeds part of the amplified energy back on the incoming line. Two adjacent repeaters would, therefore, produce *echo effects*. 4. The 21 type repeater is useful largely in cases where a single repeater is needed near the center of a uniform trunk circuit of medium length. 5. With lines of a given degree of uniformity, approximately 3 or 4 miles more amplification may be obtained with a 22 type circuit than can be secured with a 21 type circuit.

14.24 Four-Wire Repeater Circuits.—A four-wire repeater circuit is essentially the same as that of the 22 repeater circuit described above except that the attenuation or loss in the loop circuit—through which singing would have to occur—is greatly increased by making that circuit include or comprise the trunk or long distance circuit. In other words, if, in the diagram of the 22 repeater circuit, (see Fig. 7) it is assumed that the upper and lower anti-side-tone circuits and their associated amplifiers are separated by many miles—the four wires connecting these two points being the four wires of the trunk circuit—there will evidently be a greatly increased attenuation loss in the loop circuit through which the oscillating currents will have to pass. Therefore, if the increased amplification of the amplifiers used in the four-wire circuit is not greater than the increased loss in the loop circuit—due to the attenuation of the trunk—the complete circuit will have no greater tendency to sing than the corresponding 22 repeater circuit. It is, therefore, in general, possible to construct a circuit that will not sing and in which the power delivered to one line is several times greater than that which

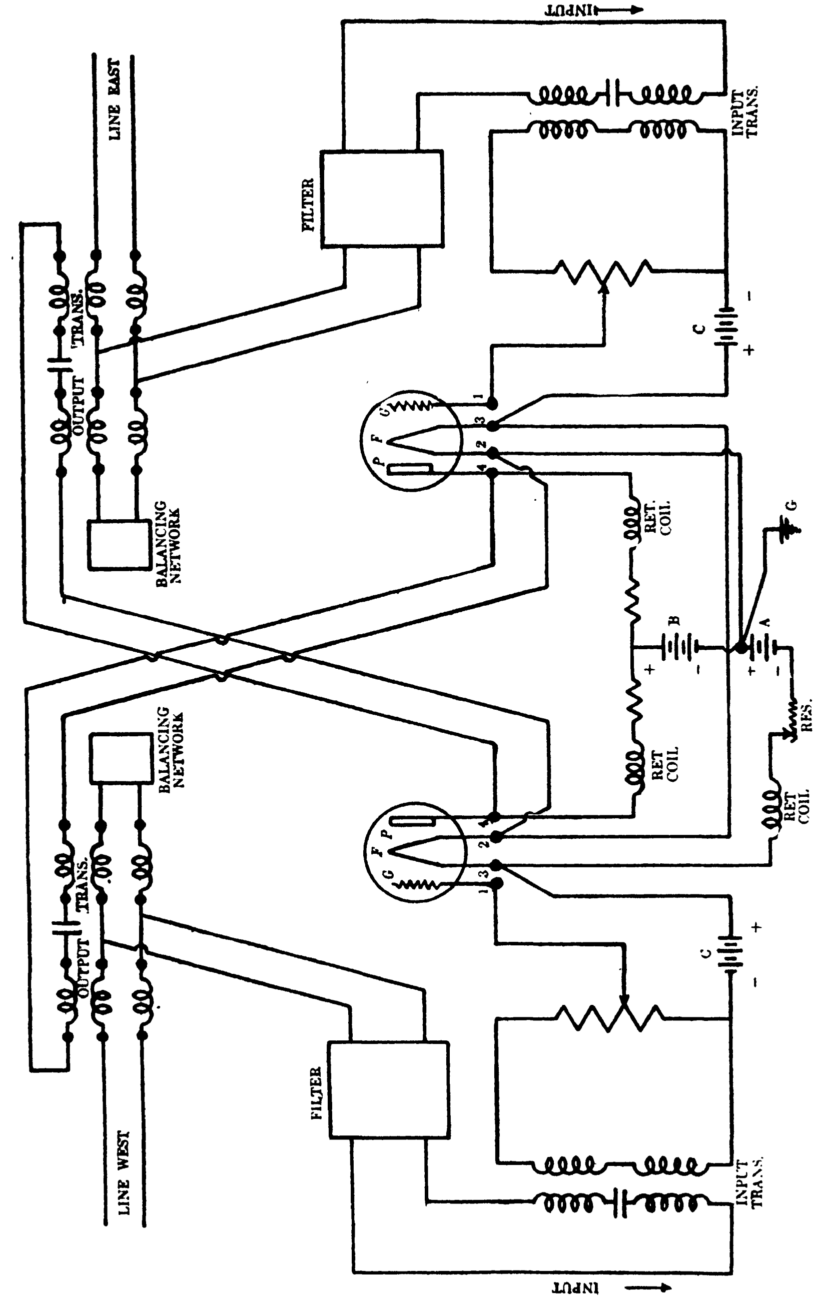

FIG. 8. Typical 22 repeater circuit employing vacuum tube repeater elements.

is impressed upon the repeater circuit from the other line. For cross-talk reasons it is not desirable in actual four-wire repeater circuits to produce all the amplification with the two terminal amplifiers but to insert a number of one-way repeaters in each of the two pairs of the four-wire circuit. A simplified drawing of such a circuit is shown in Fig. 9. The action of such a circuit is evidently analogous to that of the simple circuit without intermediate amplifiers. The spacing of the one-way repeaters and the amplification given by each of them is, as previously stated, partly determined by cross-talk considerations and partly by cost considerations.

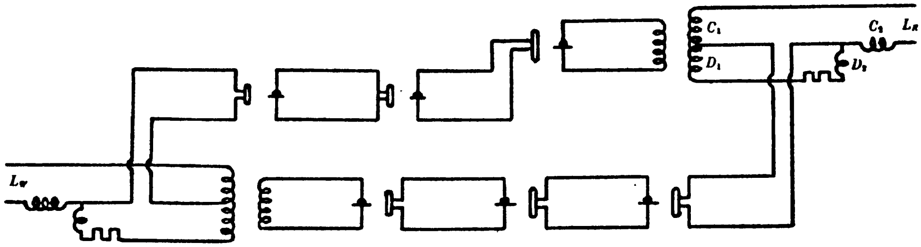

FIG. 9. Simplified diagram of a four-wire repeater circuit employing intermediate one-way repeaters.

Such a four-wire system may contain hundreds of miles of small-gauge cable with one-way repeaters introducing energy amplifications of roughly 1,000 times (30 miles of standard cable) at uniformly spaced intervals of from 50 to 100 miles. Although circuits of this type use twice as many conductors as other systems, they use smaller wires and for long distance cable operation there is a certain distance above which they are economically advantageous.

14.25 Application of Repeaters.—The successful application of repeaters to the telephone plant has completely revolutionized the methods of long-distance telephony. It is in most cases more economical to use repeaters than to load the heavier gauges of open-wire line. Furthermore, connections which formerly required open-wire conductors of the largest gauge may now often best be handled over smaller conductors of loaded cable. From the present outlook we may forecast great networks of loaded small-gauge cables equipped with repeaters. The use of heavy-gauge conductors in cables is largely a thing of the past.

On the other hand, the repeater is not a device which can be thrown in indiscriminately on any circuit and be expected to function properly. Circuits of uniform character are required, as is shown by a consideration of the requirements imposed by the balance conditions. Loading coils for use on such circuits must possess magnetic stability as well as high inductance and low resistance, and they must be quite uniformly spaced in the circuit. In general, repeatered telephone circuits call for a higher grade of plant maintenance and for special testing methods and equipment.*

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* For a more detailed discussion of some of the difficulties encountered in repeatered lines, as well as for a general view of some of the other problems of telephone transmission over long circuits, reference may be made to an article by J. J. Pilliod entitled "Philadelphia-Pittsburgh Section of the New York-Chicago Cable," *Journal of the A. I. E. E.*, Vol. XLI, August, 1922.

CHAPTER XV

RESONANT AND ANTI-RESONANT CIRCUITS

15.0 The study of resonant and anti-resonant circuits is not only interesting but is also very important for a clear understanding of the more complex subject of *Selective Networks* (see Chapter XVI). The simple type of resonant circuit, containing inductance, capacity, and resistance all in series, is well known. *Resonance is said to occur in such a circuit when the frequency is such that the positive reactance of the inductance is equal and opposite to the negative reactance of the condenser, i.e., when the impedance is a minimum.*

When one considers parallel circuits, however, consisting of a condenser in parallel with an inductance—the latter being in series with a resistance—the problem of determining *anti-resonance* becomes somewhat more complicated. Consider the circuit shown in Fig. 1, and assume that we wish to determine the proper value of the capacity to give *anti-resonance* or *to produce a maximum value of the absolute magnitude of the impedance Z of the combination.* If the condenser is assumed to have a capacity of C farads (and negligible resistance) and the coil to have an inductance of L henrys and an effective resistance of R ohms, it is easy to show that the impedance Z is

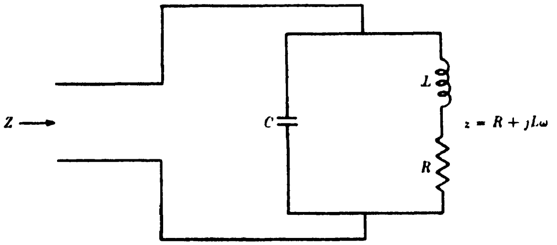

FIG. 1. Circuit containing a condenser in parallel with a combination consisting of an inductance and resistance in series.

$$Z = \frac{R + j\omega[L - C(R^2 + L^2\omega^2)]}{R^2C^2\omega^2 + (LC\omega^2 - 1)^2} \tag{1}$$

The absolute value of Z is

$$|Z| = \sqrt{\frac{R^2 + L^2\omega^2}{R^2C^2\omega^2 + (LC\omega^2 - 1)^2}} = \frac{|z|}{\sqrt{R^2C^2\omega^2 + (LC\omega^2 - 1)^2}} \tag{2}$$

Hence, the ratio r of the absolute value of Z to that of $|R+jL\omega|$ or $|z|$ is

$$r \equiv \left|\frac{Z}{z}\right| = \frac{1}{\sqrt{R^2C^2\omega^2 + (LC\omega^2 - 1)^2}} \tag{3}$$

If we let $Q \equiv L\omega/R$ (Q being called for convenience the *coil dissipation constant*), (3) becomes

$$r = \frac{1}{\sqrt{R^2C^2\omega^2 + (QRC\omega - 1)^2}} \tag{4}$$

By differentiation it can be shown that, if C is variable, the value of r given by (4) is a maximum when

$$C = \frac{Q}{R\omega(1+Q^2)} = \frac{L}{R^2 + L^2\omega^2} = \frac{L}{|z^2|} \tag{5}$$

Putting this value in (4),

$$r = \sqrt{1+Q^2} \tag{6}$$

But $Q = \tan\theta$, where θ is the phase angle of the combination consisting of the resistance and inductance. Hence

$$r = \sqrt{1+\tan^2\theta} = \sec\theta \tag{7}$$

It is also easy to show that, when (5) holds, the phase angle of Z, which will be called ϕ, is zero. Hence, at anti-resonance (that is, if the capacity C is assumed to be varied until $|Z|$ is a maximum)

$$Z_{\text{MAX.}} = |z| \sec\theta \,/0^\circ = \left|\frac{z}{\cos\theta}\right| /0^\circ = \left|\frac{z^2}{z\cos\theta}\right| \underline{/0^\circ} = \left|\frac{z^2}{R}\right| \underline{/0^\circ} \tag{8}$$

This equation then furnishes a ready method of finding the maximum impedance, $Z_{\text{MAX.}}$, which will result if the capacity is varied until anti-resonance is obtained. As an example of the above, suppose we had a coil whose impedance $|z|\,/\theta$ at a given frequency was $100\,\underline{/85^\circ}$ ohms and wished to know what would be the maximum impedance that could be obtained, at the given frequency, by shunting the coil with a condenser. From (8) it is apparent that

$$Z_{\text{MAX}} = \frac{100}{.087} = 1{,}150\,/0^\circ \tag{9}$$

If in the above, $\omega = 5{,}000$, L will be .01992 henry and R will be 8.72 ohms. Whence, from equation (5), $C = 1.992$ mf.

Suppose that, instead of assuming C to vary until anti-resonance occurred, the inductance L had been assumed to be the variable. With

L as a variable, it is found by differentiation of (2) that $|Z|$ is a maximum when

$$L = \frac{1 + \sqrt{1 + 4R^2C^2\omega^2}}{2C\omega^2} \tag{10}$$

Similarly, if the frequency is assumed to be the variable, it is found from differentiation of (2) that the absolute value of the impedance $|Z|$ is a maximum when

$$\omega = 2\pi f = \sqrt{\frac{\sqrt{1 + 2R^2\left(\frac{C}{L}\right)}}{LC} - \left[\frac{R}{L}\right]^2} \tag{11}$$

From the above equations, (5), (10), and (11), it is apparent that the condition for anti-resonance—when there is resistance in the coil—is different for all three variables C, L and ω, and hence in using the term *anti-resonance* when referring to a parallel circuit containing resistance as well as capacity and inductance, it is essential that the quantity assumed to be the variable should always be mentioned. In many cases the results obtained, by assuming the different elements as variables, are not materially different. For example, in the numerical case previously taken, i.e., $L = .01992$ henry, $R = 8.72$ ohms and $C = 1.992$ mf., if the frequency is assumed to be the variable, equation (11) gives $\omega = 5{,}020$. Whence, the point of anti-resonance is a little over three cycles higher than that obtained when C was considered to be the variable.

Unless otherwise stated, it is customary to assume that the *frequency* is the variable quantity referred to in problems involving resonance or anti-resonance.

Resonant circuits have several peculiar characteristics which would not be expected to exist—unless the mathematical proof of them were readily available. Consider, for example, the circuit shown in Fig. 2.

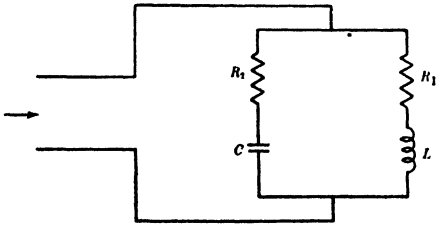

Fig. 2. Structure comprising a resistance and condenser, in parallel with a resistance and inductance, and having an impedance whose value is a constant pure resistance at all frequencies.

The impedance Z of this parallel combination is evidently

$$Z = R + jX = \frac{(R_1 + jL\omega)\left(R_2 + \dfrac{1}{jC\omega}\right)}{(R_1 + jL\omega) + \left(R_2 + \dfrac{1}{jC\omega}\right)} \tag{12}$$

Keeping in mind that in the algebra of complex quantities (see formula (15) of Appendix B)

$$\frac{A + jB}{C + jD} = \frac{AC + BD}{C^2 + D^2} + j\frac{BC - AD}{C^2 + D^2} \tag{13}$$

it is easy to show from (12) that

$$R = \frac{(R_1 + R_2)\left(R_1R_2 + \dfrac{L}{C}\right) + \left(R_2L\omega - \dfrac{R_1}{C\omega}\right)\left(L\omega - \dfrac{1}{C\omega}\right)}{(R_1 + R_2)^2 + \left(L\omega - \dfrac{1}{C\omega}\right)^2} \tag{14}$$

and

$$X = \frac{(R_1 + R_2)\left(R_2L\omega - \dfrac{R_1}{C\omega}\right) - \left(R_1R_2 + \dfrac{L}{C}\right)\left(L\omega - \dfrac{1}{C\omega}\right)}{(R_1 + R_2)^2 + \left(L\omega - \dfrac{1}{C\omega}\right)^2} \tag{15}$$

If the impedance of the parallel combination is to be a pure resistance and independent of the frequency, it is evident that the reactance X must be zero. Equating then the numerator of the expression for X to zero, it is seen that the reactance X will be zero if

$$L\omega\left(R_2^2 - \frac{L}{C}\right) + \frac{1}{C\omega}\left(\frac{L}{C} - R_1^2\right) = 0 \tag{16}$$

This relation will evidently hold if

$$R_1 = R_2 = \sqrt{\frac{L}{C}} \tag{17}$$

Hence, if the above relation is met, the impedance Z will be a pure resistance and independent of the frequency. By substituting the values of (17) in (14), the value of the impedance Z when (17) is fulfilled is seen to be

$$Z = R_1 = R_2 = \sqrt{\frac{L}{C}} \tag{18}$$

This is but a special case of a *constant resistance structure* (for general relations see Sect. I of Appendix D), the general characteristics of which are discussed in detail in Sect. 18.1.

Another interesting example of a circuit consisting of inductance, capacity, and resistance, which is somewhat similar to the circuit just considered, is the bridge circuit shown in Fig. 3. If, in such a circuit $R = \sqrt{L/C}$, it can easily be demonstrated that the impedance looking into the circuit is $Z = R = \sqrt{L/C}$ and is entirely independent of thefrequency. It is to be noted that the relation given above, which is necessary to make the impedance of the circuit independent of the frequency, is essentially identical to that (see equation (17)) required in the simpler circuit previously discussed. The generality of this relation may also readily be seen by reference to formulae 29 of Appendix D.

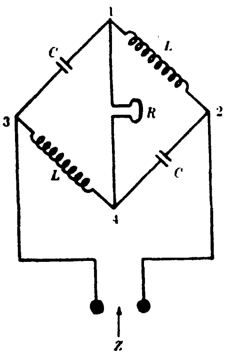

FIG. 3. Bridge structure, composed of inductances and capacities, whose impedance is a constant pure resistance at all frequencies.

If, in the above circuit there is mutual inductance M between the two windings—in such a direction that 1–2–3–4 is a series aiding connection, it can be shown that the impedance Z will be independent of the frequency if

$$R = \sqrt{\frac{L + M}{C}} \tag{19}$$

In this case Z will have a value similar to that obtained before, viz.:

$$Z = R = \sqrt{\frac{L + M}{C}} \tag{20}$$

Resonant circuits have many other very interesting properties. One property which has been used commercially to a certain extent is the conversion of a constant potential circuit into a constant current circuit, or vice versa. Referring to the circuit shown in Fig. 4, the equation for the received current I_R is

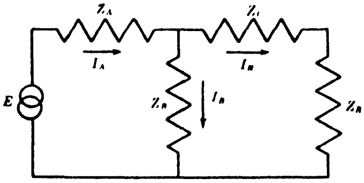

FIG. 4. *T* network form of circuit for converting a constant potential into a constant current.

$$I_R = \frac{EZ_B}{Z_AZ_B + (Z_A + Z_B)(Z_C + Z_R)} \tag{21}$$

If Z_A and Z_B are pure reactances and if their sum is equal to zero, the denominator of this equation will reduce to $Z_A Z_B$ and the received current under these conditions becomes

$$I_R = \frac{E}{Z_A} \tag{22}$$

Hence, if the impedance Z_A is a pure reactance and of the value $1/jC\omega$ (or $jL\omega$) and if the impedance Z_B is likewise a pure reactance and has the value $jL\omega$ (or $1/jC\omega$) and if $LC\omega^2 = 1$, it is evident that (22) will hold and that the received current in such a circuit will be

$$I_R = \frac{E}{jL\omega} \qquad \text{or} \qquad I_R = -jEC\omega \tag{23}$$

It is then apparent that, in such a case, the received current I_R is independent of the value of the load impedance. Such a circuit will, therefore, at any given frequency, produce a constant current in the received impedance regardless of the magnitude of this impedance. Such circuits have been used commercially in connection with series arc light systems where the resistance of the circuit varies from time to time and where it is desired to keep a very constant current flowing in the circuit.

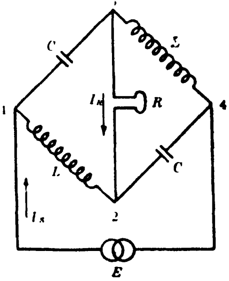

FIG. 5. Bridge form of circuit for converting a constant potential into a constant current.

Another typical example of a resonant circuit which may be used for converting a constant potential circuit into a constant current circuit is that shown in Fig. 5. If there is assumed to be a mutual inductance M between the two windings and if the windings are so wound that 1–2–3–4 is a series aiding connection, it can be shown, if the frequency is such that

$$(L + M)C\omega^2 = 1 \tag{24}$$

that the current in the receiving impedance is independent of the value of this impedance and is equal to

$$I_R = \frac{jE}{(L + M)\omega} \tag{25}$$

CHAPTER XVI

THEORY OF WAVE FILTERS

16.0 A *selective network* is defined as a structure which is designed to give a predetermined transmission loss characteristic. If the structure is to have a negligible transmission loss over a certain definite range or ranges of frequencies and is to have an appreciable loss at all other frequencies the structure is called a *wave filter*. On the other hand, if the structure is to have a loss characteristic that is more or less gradual or varied in its slope, it is often referred to either as a *corrective network* or as a *simulative network*.* It is obvious that the main distinction which is made between the terms *wave filter* and *corrective network* is that in the former there is a sharp line of demarcation between the attenuated and the unattenuated frequency ranges, whereas no such sharp line exists in the corrective network.

Electric wave filters were invented by Campbell † and have a very large number of practical applications, some of the more important of which are as follows:

A-C. Oscillators. To eliminate harmonics from oscillators and thereby obtain a purer sine wave than would otherwise be possible.

D-C. Generators. To suppress commutator ripples in circuits supplying direct current from d-c. generators.

Composite Sets. To separate the currents of telephone and telegraph (d-c. or carrier) circuits.

Radio Operation. To separate the received and transmitted signals when a single antenna is used for both transmitter and receiver. To eliminate the carrier and one side-band, thereby enabling all of the output power to be put into the other side-band. To make possible the separation of two incoming telephone messages involving currents of approximately the same frequencies.

Carrier Current Telephony and Telegraphy Over Wires. To separate and confine currents of the various frequencies in their proper channels.‡

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* Simulative networks include such apparatus as artificial lines or cables.

† See U. S. patent No. 1,227,113. For a discussion of the "Physical Theory of the Electric Wave Filter," by G. A. Campbell, see *The Bell System Technical Journal*, Vol. I, No. 2, Nov. 1922.

‡ For a detailed explanation of the use of filters in carrier current telephony, see paper by E. H. Colpitts and O. B. Blackwell on "Carrier Current Telephony and Telegraphy," *Transactions of the A. I. E. E.*, Vol. XL (1921), pp. 205–296.

Study of Speech. To eliminate, to any desired extent, any given frequencies (in an otherwise distortionless telephone system) in laboratory investigations of telephone quality and of the composition of speech.

Submarine Signalling. To eliminate disturbing sounds from water noises, waves, etc., that are encountered in various schemes employed for submarine signalling and thus to give a clearer tone to the signal and effectively to increase the range of transmission.

16.1 **General Types of Wave Filters.**—In view of the widely different characteristics of various types of wave filters it has been commonly agreed to distinguish the more frequently used types by the following designations:

1. Low Pass (L.P.) Wave Filters.
2. High Pass (H.P.) Wave Filters.
3. Band Pass (B.P.) Wave Filters.
4. Band Elimination (B.E.) or Low and High (L. & H.P.) Pass Wave Filters.

A *low pass wave filter* is a structure which readily passes currents of all frequencies from zero up to a certain frequency (called the cut-off frequency, f_c), and which effectually bars currents of all higher frequencies.

A *high pass wave filter* is one which passes currents of all frequencies from infinity down to a certain frequency (also called the cut-off frequency, f_c), and effectually bars currents of all lower frequencies.

A *band pass wave filter* is one which readily passes currents that lie between two cut-off frequencies (f_1 and f_2), and effectually bars currents of all frequencies outside of this range.

Finally, a *band elimination or a low and high pass wave filter* is a structure which effectually bars currents that lie between two cut-off frequencies (f_1 and f_2) and readily passes currents of all frequencies outside of this range.

It is evident from a little consideration that we can easily construct a band pass or a band elimination filter from two filters—one of the low pass type and the other of the high pass type. For example, if a high pass and a low pass wave filter are in series with each other, as shown in Fig. 1, the combination will, in effect, act as a band pass wave filter that will

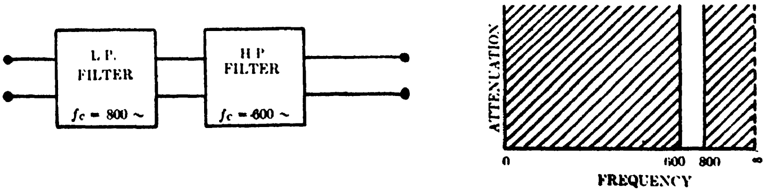


FIG. 1. Band pass filter formed by a high pass filter in series with a low pass filter

pass currents between the cut-off frequencies of the two filters—provided the cut-off frequency of the low pass filter is higher than that of the high pass filter. In the specific case indicated the structure will act as a band pass wave filter which will pass only those currents which have a frequency lying between 600 and 800 cycles.

Similarly, considering the circuit of Fig. 2, in which a high pass filter is shown in parallel with a low pass filter—the latter filter having the lower cut-off frequency—the combination is in effect a band elimination filter which will pass only currents of those frequencies which are above the cut-off point of the high pass wave filter and currents of those frequencies that are below the cut-off point of the low pass wave filter. In the specific case indicated, currents of those frequencies which lie between 600 and 800 cycles would be attenuated.

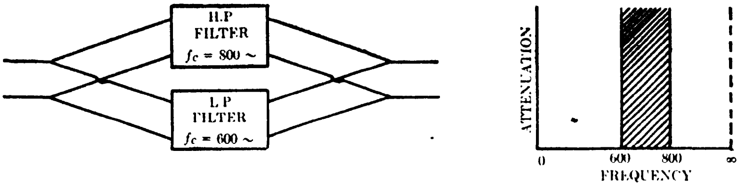
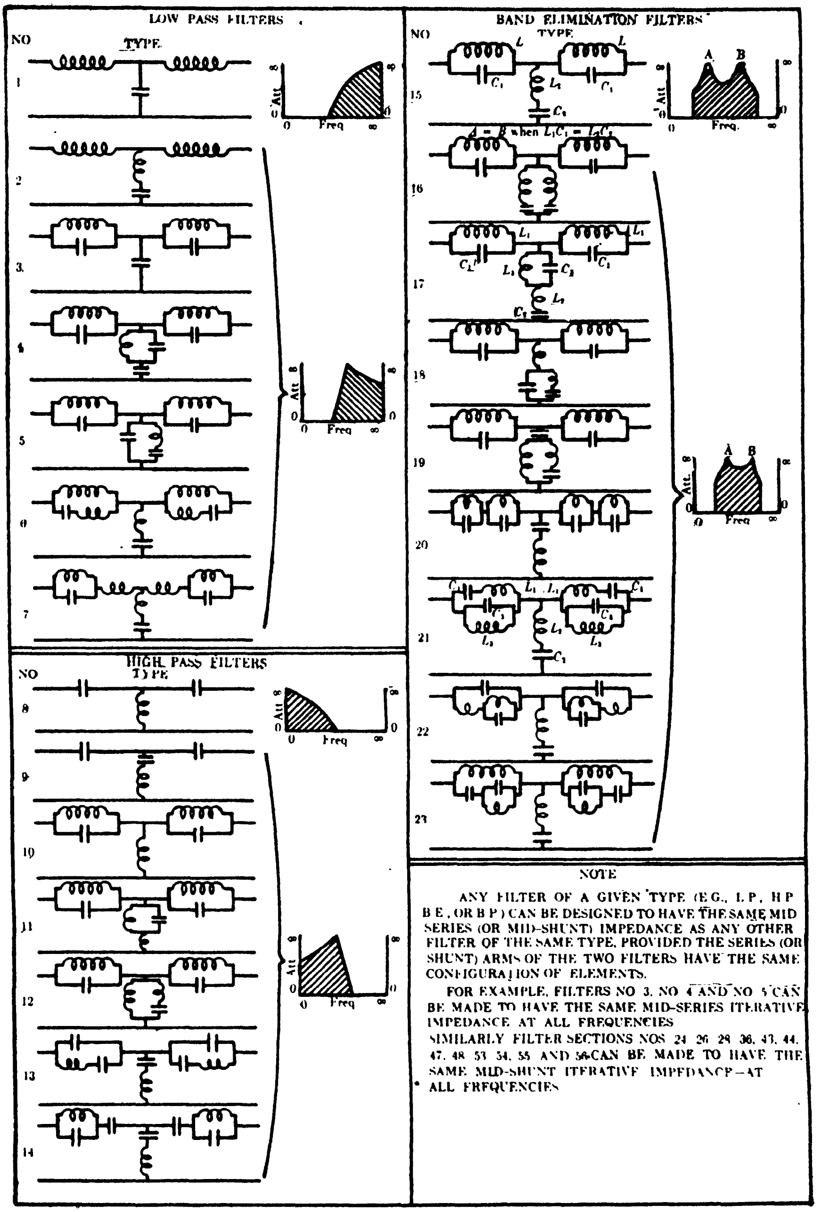


FIG. 2. Band elimination filter formed by a high pass filter connected in parallel with a low pass filter.

16.11 Lists of Ideal Wave Filter Structures.—If a filter structure is to be *ideal,* that is, if it is to have a zero attenuation constant over a certain frequency range, there can be no resistances in the filter. In other words, the ideal or non-dissipative wave filter must consist entirely of pure reactances. It is furthermore true that, in general, almost every symmetrical combination of condensers and inductances may be regarded as a section of an ideal wave filter. Ordinarily, however, one is only interested in those combinations of reactances which give a single (or at most a double) transmitted band of frequencies. This limits our interest and consideration to a relatively small number of structures, or combinations of reactances, the most important of which are as follows.

Filter sections of the *ladder type*—that is, *the type of structure in which a certain combination of reactances is directly in series and another combination is directly in shunt with the line*—are listed in Fig. 3. This figure gives a list of the single band structures (L.P., H.P., B.P. and B.E.) of the ladder or *series-shunt* type—on the assumption that there is no mutual inductance between any of the coils. It also gives the general shapes of the attenuation characteristics of the various structures. The method of obtaining these attenuation characteristics as well as that of



NOTE

ANY FILTER OF A GIVEN TYPE (E.G., L.P., H.P. B.E., OR B.P.) CAN BE DESIGNED TO HAVE THE SAME MID SERIES (OR MID-SHUNT) IMPEDANCE AS ANY OTHER FILTER OF THE SAME TYPE, PROVIDED THE SERIES (OR SHUNT) ARMS OF THE TWO FILTERS HAVE THE SAME CONFIGURATION OF ELEMENTS.

FOR EXAMPLE, FILTERS NO 3, NO 4 AND NO 5 CAN BE MADE TO HAVE THE SAME MID-SERIES ITERATIVE IMPEDANCE AT ALL FREQUENCIES

SIMILARLY FILTER SECTIONS NOS 24 26 29 36, 43, 44, 47, 48 53 54, 55 AND 56 CAN BE MADE TO HAVE THE SAME MID-SHUNT ITERATIVE IMPEDANCE—AT ALL FREQUENCIES

FIG. 3. A list of wave filter sections of the ladder type, having not more than four reactance elements in any one arm, and which, by proper design, have only a single transmission or attenuation band.

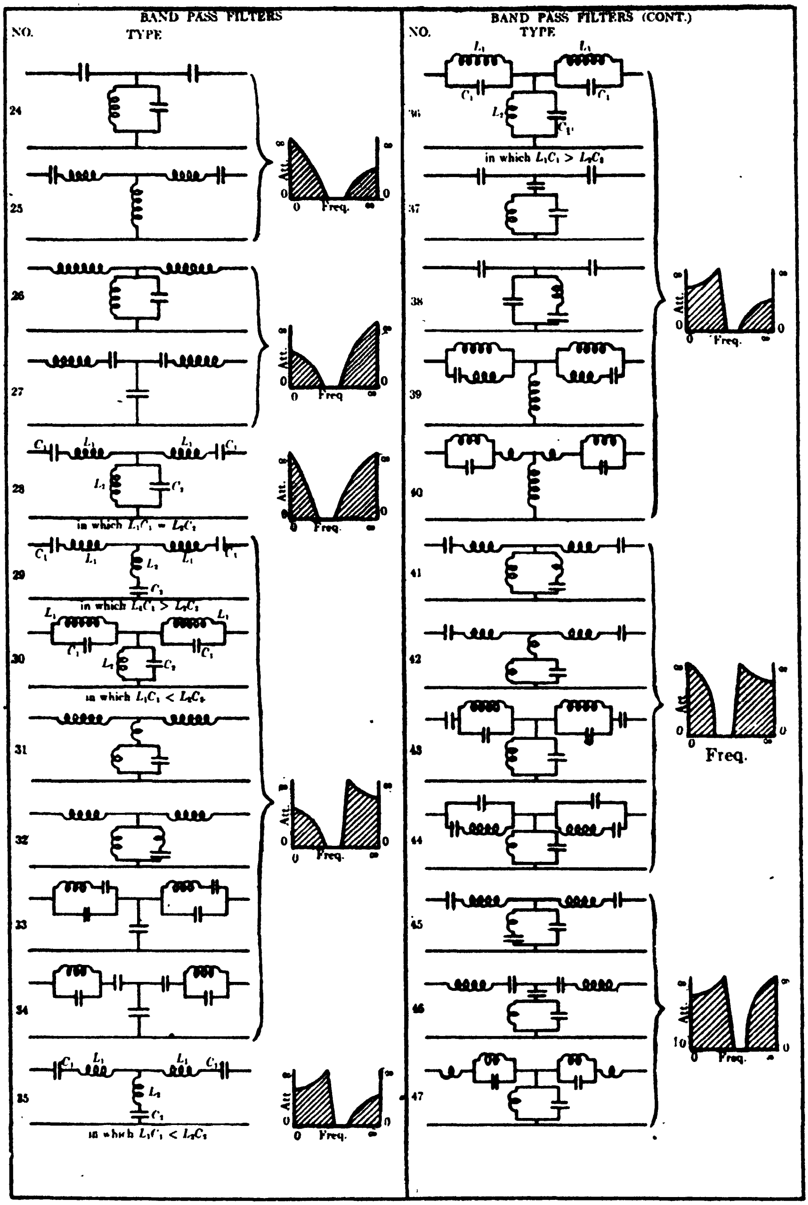
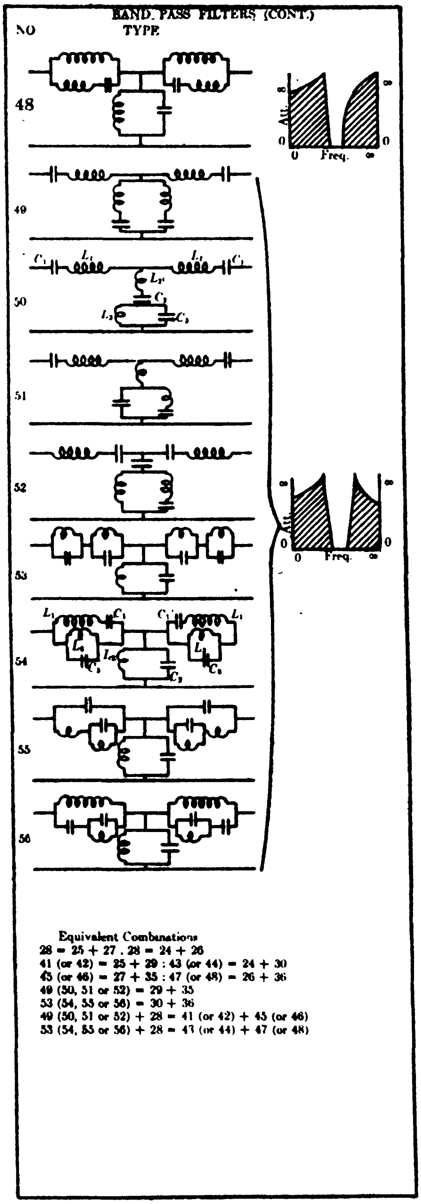

BAND PASS FILTERS
NO.
TYPE
24
25
26
27
28
in which $L_1C_1 = L_2C_2$
29
in which $L_1C_1 > L_2C_2$
30
in which $L_1C_1 < L_2C_2$
31
32
33
34
35
in which $L_1C_1 < L_2C_2$
BAND PASS FILTERS (CONT.)
NO.
TYPE
36
in which $L_1C_1 > L_2C_2$
37
38
39
40
41
42
43
44
45
46
47
L_1
L_2
C_1
C_2
Att.
Freq.
0
∞

FIG. 3 (*Continued*).



Equivalent Combinations
28 = 25 + 27 . 28 = 24 + 26
41 (or 42) = 25 + 29 : 43 (or 44) = 24 + 30
45 (or 46) = 27 + 35 : 47 (or 48) = 26 + 36
49 (50, 51 or 52) = 29 + 35
53 (54, 55 or 56) = 30 + 36
49 (50, 51 or 52) + 28 = 41 (or 42) + 45 (or 46)
53 (54, 55 or 56) + 28 = 43 (or 44) + 47 (or 48)

FIG. 3 (*Continued*).

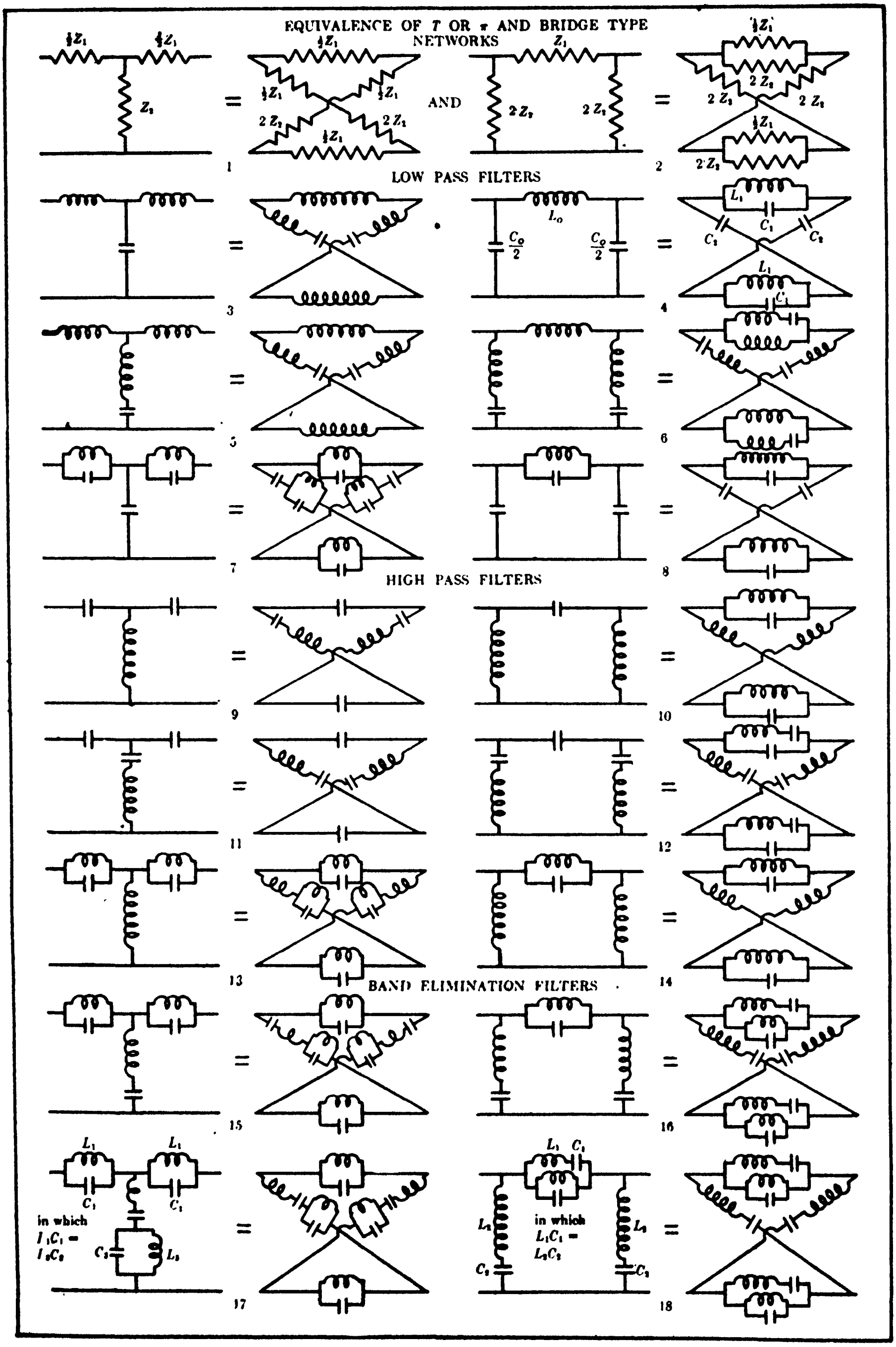


FIG. 4. List of single band wave filter sections of the bridge type together with the equivalent ladder type of structures.

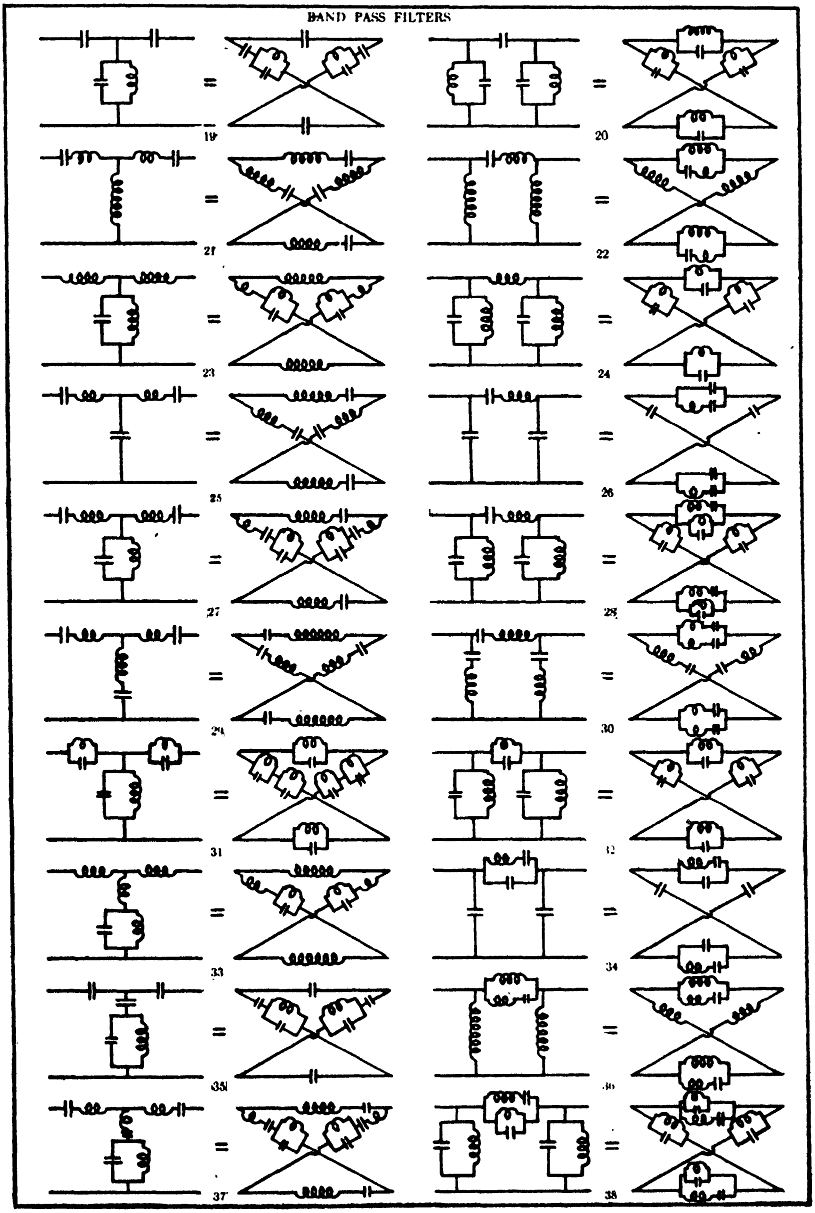


FIG. 4 (*Continued*).

obtaining all of the necessary formulae applying to the various structures will be described later.

A list of some of the more important structures of the so-called *lattice* or *bridge* type may be found in Fig. 4. This figure also shows the equivalence of the lattice type of structures with those of the ladder type.

There is still another type of filter structure which should be given some consideration and that is the so-called *bridged-T* type of structure. A list of some of the more important structures of this type together with the equivalent ladder type of structures is given in Fig. 5. The equivalence of the various *bridged-T* structures to the corresponding ladder type of structures is obvious when T networks of condensers or inductances are replaced by Π networks or vice versa.

16.2 Propagation Characteristics of a Non-Dissipative Symmetrical Structure.—Since a wave filter is, in the ideal case, a structure that has no dissipation, it is desirable to consider, before going further, what are the propagation characteristics of such a structure. In Fig. 6, it is evident from formula (7) of Chapter XI that:

$$P \equiv A + jB = 2 \sinh^{-1} \frac{1}{2} \sqrt{\frac{Z_1}{Z_2}} \tag{1}$$

If the series and shunt impedances Z_1 and Z_2 are—as they would be in the case of an ideal filter—pure reactances, the quantity Z_1/Z_2 must evidently have either a zero phase angle or a 180° phase angle. Hence, Z_1/Z_2 will either be a real positive or a real negative quantity and cannot be a complex quantity.

16.21 When Z_1/Z_2 is Positive.—If Z_1/Z_2 is a positive quantity equation (1) gives:

$$P = A + jB = 2 \sinh^{-1} \frac{1}{2} \sqrt{\frac{Z_1}{Z_2}} + jO \tag{2}$$

or

$$A = 2 \sinh^{-1} \frac{1}{2} \sqrt{\frac{Z_1}{Z_2}} \qquad \text{and} \qquad B = 0 \tag{3}$$

Under these conditions, there will then always be attenuation in the structure.

16.22 When Z_1/Z_2 is Negative.—Under these conditions equation (1) states that:

$$\sinh\left(\frac{A}{2} + j\frac{B}{2}\right) = \frac{1}{2} \sqrt{\frac{Z_1}{Z_2}} \tag{4}$$

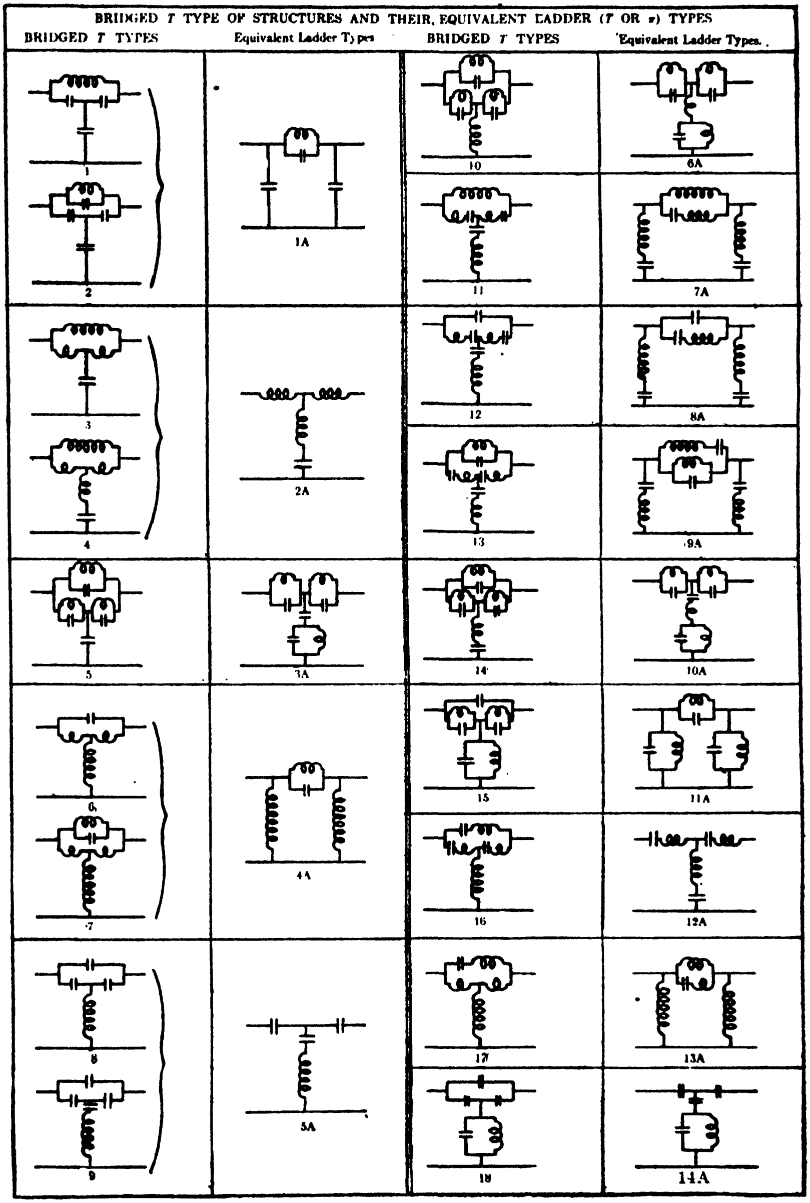


FIG. 5. List of single band wave filter structures of the bridged-*T* type and their equivalent *T* or Π networks.

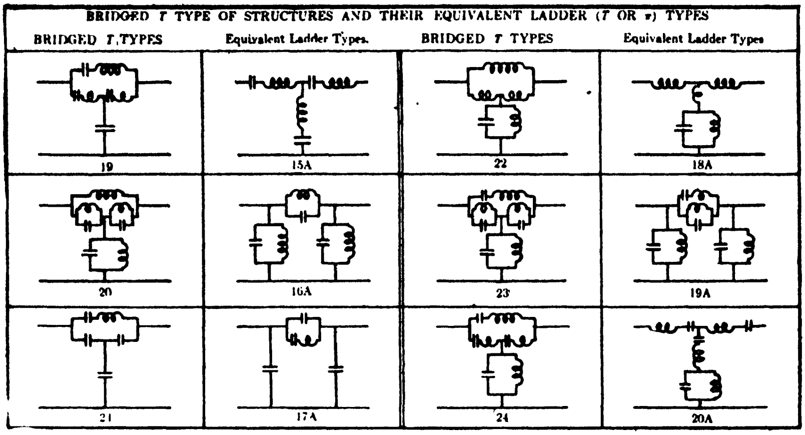


FIG. 5 (*Continued*).

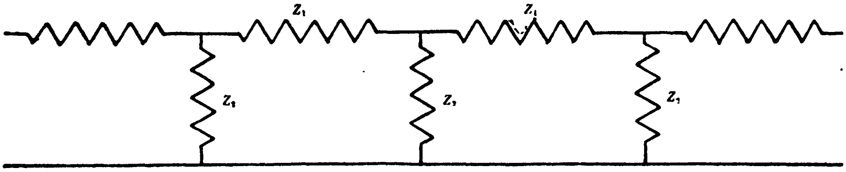


FIG. 6. Symbolic structure of the ladder type.

But it is well known (see equation (76) of Appendix B) that

$$\sinh(u + jv) = (\sinh u \cos v) + j(\cosh u \sin v) \tag{5}$$

Hence

$$\sinh\left(\frac{A}{2} + j\frac{B}{2}\right) = \sinh\frac{A}{2}\cos\frac{B}{2} + j\cosh\frac{A}{2}\sin\frac{B}{2} = \frac{1}{2}\sqrt{\frac{Z_1}{Z_2}} \tag{6}$$

But since Z_1/Z_2 is negative, the expression $1/2\sqrt{Z_1/Z_2}$ must be a pure imaginary. Therefore, from equation (6) either

$$\sinh\frac{A}{2} = 0 \qquad \text{or} \qquad \cos\frac{B}{2} = 0$$

16.221 When $|Z_1/Z_2|$ Lies Between Zero and Four.—Assume that $\sinh A/2 = 0$. Then

$$A = 0 \qquad \text{and} \qquad \cosh\frac{A}{2} = 1 \tag{7}$$

or from (6)

$$j\sin\frac{B}{2} = \frac{1}{2}\sqrt{\frac{Z_1}{Z_2}} \tag{8}$$

or

$$\sin\frac{B}{2} = \frac{1}{2}\sqrt{-\frac{Z_1}{Z_2}} \qquad \text{or} \qquad B = 2\sin^{-1}\frac{1}{2}\sqrt{-\frac{Z_1}{Z_2}} \tag{9}$$

In this last expression, it can be shown (by assuming a slight amount of dissipation in Z_1 and Z_2) that the angle whose sine is $1/2\sqrt{-Z_1/Z_2}$ is to be taken in the first quadrant if Z_1 is a positive and in the second quadrant if Z_1 is a negative reactance. It should be pointed out that equation (9) is useful only when $|Z_1/Z_2|$ is less than 4.

16.222 When $|Z_1/Z_2|$ is Greater than Four.—Assume in equation (6) that

$$\cos\frac{B}{2} = 0$$

Then

$$\frac{B}{2} = (2K - 1)\frac{\pi}{2}$$

where K is an integer; or

$$B = (2K - 1)\pi \tag{10}$$

Consequently

$$\sin\frac{B}{2} = \pm 1$$

or from (6)

$$\pm j\cosh\frac{A}{2} = \frac{1}{2}\sqrt{\frac{Z_1}{Z_2}} \qquad \text{or} \qquad \cosh\frac{A}{2} = \frac{1}{2}\sqrt{-\frac{Z_1}{Z_2}} \tag{11}$$

Whence

$$\frac{A}{2} = \cosh^{-1}\frac{1}{2}\sqrt{-\frac{Z_1}{Z_2}} \qquad \text{or} \qquad A = 2\cosh^{-1}\frac{1}{2}\sqrt{-\frac{Z_1}{Z_2}} \tag{12}$$

16.23 Summary of Propagation Characteristics of Non-Dissipative Symmetrical Structures.—From the foregoing formulae it is evident that the following summary may be given:

16.231 Attenuation Range.—If Z_1/Z_2 is *positive:*

$$P = A + jB = 2\sinh^{-1}\frac{1}{2}\sqrt{\frac{Z_1}{Z_2}} + jO = 2\cosh^{-1}\sqrt{1 + \frac{1}{4}\frac{Z_1}{Z_2}} + jO \tag{13}$$

If Z_1/Z_2 *is negative and is greater, in absolute magnitude, than* 4:

$$\begin{aligned} P = A + jB &= 2\cosh^{-1}\frac{1}{2}\sqrt{-\frac{Z_1}{Z_2}} + j(2K - 1)\pi \\ &= 2\sinh^{-1}\sqrt{-1 - \frac{1}{4}\frac{Z_1}{Z_2}} + j(2K - 1)\pi \end{aligned} \tag{14}$$

16.232 Non-Attenuation Range.—If Z_1/Z_2 *is negative and is less, in absolute magnitude, than* 4:

$$P = A + jB = O + j2\sin^{-1}\frac{1}{2}\sqrt{-\frac{Z_1}{Z_2}} = O + j2\cos^{-1}\sqrt{1 + \frac{1}{4}\frac{Z_1}{Z_2}} \tag{15}$$

in which the angle, whose sine is $1/2\sqrt{-Z_1/Z_2}$, is taken in the first quadrant if the reactance Z_1 is positive and in the second quadrant if Z_1 is negative. Similarly, the angle whose cosine is $\sqrt{1 + Z_1/4Z_2}$, is to be taken in the first quadrant if Z_1 is a positive reactance and in the fourth quadrant if Z_1 is a negative reactance.

From the preceding formulae it is seen that non-dissipative recurrent structures of the ladder or series-shunt type will pass freely only those currents which are of such frequencies that the ratio of the total series impedance (between adjacent shunt arms) to the shunt impedance lies between zero and -4. In other words, *a non-dissipative recurrent structure of the ladder type having series impedances Z_1 and shunt impedances Z_2 will pass readily only currents of such frequencies as will make the value of the ratio Z_1/Z_2 lie between 0 and -4.*

16.3 *M*-Derived Types of Structures.—An extremely simple and useful method of obtaining the formulae for the propagation constant and the iterative impedance of many complex types of symmetrical wave filter structures is by considering such structures to be *derived* from the simpler types.* The simple types from which the more complex structures are derived are called *prototypes*—the more complex structures being called either the *m-derived types* or simply *m-types.*

Derived types of structures are of two varieties: (1) the *mid-series equivalent derived type,* in which the mid-series iterative impedance of the derived type is exactly the same as the mid-series iterative impedance of the prototype; and (2) the *mid-shunt equivalent derived type,* in which the mid-shunt iterative impedance of the derived type is identical with the mid-shunt iterative impedance of the prototype.

16.31 Mid-Series Equivalent *M*-Derived Type.—Consider two periodic structures of the types shown in Figs. 7A and 7B, in which the

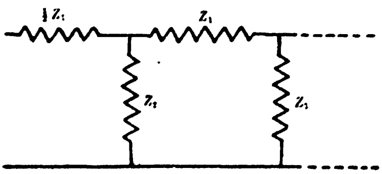


FIG. 7A. Prototype.

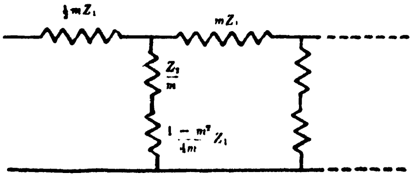


FIG. 7B. Mid-series equivalent *m*-derived type.

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* This method of "deriving" structures—as well as many of the other schemes discussed in this chapter—was first pointed out by O. J. Zobel. See Zobel's paper on the "Theory and Design of Uniform and Composite Electric Wave-Filters," in *The Bell System Technical Journal,* Vol. II, No. 1, Jan. 1923.

structure shown in Fig. 7A is regarded as the *prototype* and that in Fig. 7B as the derived or *m-type.*

From equation (1) in Chapter XI, the mid-series iterative impedance of the structure shown in Fig. 7A is:

$$Z_K = \sqrt{Z_1 Z_2 \left(1 + \frac{1}{4}\frac{Z_1}{Z_2}\right)} \tag{16}$$

Similarly, the mid-series iterative impedance of the structure shown in Fig. 7B is:

$$\begin{aligned} Z_K &= \sqrt{\left[mZ_1\right]\left[\frac{Z_2}{m} + \frac{1 - m^2}{4m} Z_1\right]\left[1 + \frac{1}{4}\frac{mZ_1}{\frac{Z_2}{m} + \frac{1 - m^2}{4m} Z_1}\right]} \\ &= \sqrt{Z_1 Z_2 \left(1 + \frac{1}{4}\frac{Z_1}{Z_2}\right)} \end{aligned} \tag{17}$$

In other words, the mid-series iterative impedances of the two structures are identical.

The propagation constant of any symmetrical structure is, as has been seen, dependent only upon the *ratio* of the impedance of the series arm to that of the shunt arm; that is, in Fig. 7A upon Z_1/Z_2. In the derived type, however, which is shown in Fig. 7B this ratio is evidently:

$$\left(\frac{Z_1}{Z_2}\right)_m \equiv \frac{mZ_1}{\frac{Z_2}{m} + \frac{1 - m^2}{4m} Z_1} = \frac{4m^2\left(\frac{Z_1}{Z_2}\right)}{(1 - m^2)\left(\frac{Z_1}{Z_2}\right) + 4} \tag{18}$$

It will be noted from this last expression that when Z_1/Z_2 of the prototype is zero, then the corresponding ratio $(Z_1/Z_2)_m$ of the derived type is also zero. Similarly, when in the prototype $Z_1/Z_2 = -4$, the value of $(Z_1/Z_2)_m$ for the derived *m-type* is also seen to be equal to -4. Hence, the derived type as shown in Fig. 7B has identically the same cut-off frequencies, etc., as the prototype from which it is derived.

It will be noticed that the denominator of the last expression in (18) is zero when:

$$\frac{Z_1}{Z_2} = \frac{-4}{1 - m^2} \qquad \text{or when} \qquad m = \sqrt{1 + \frac{4Z_2}{Z_1}} \tag{19}$$

In other words, under the above condition the ratio, $(Z_1/Z_2)_m$, of the series to the shunt impedances of the derived *m-type* structure becomes infinite and hence, the derived structure has infinite attenuation when the above condition is fulfilled.

It is, therefore, seen that it is always possible, by the relations shown in Figs. 7A and 7B, to derive a structure from a prototype which will have the same mid-series iterative impedance and the same cut-off frequencies as the prototype but which will, in general, have a different propagation constant—the attenuation constant of the derived type always having a point of infinite value when the condition given in equation (19) is met.

16.32 Mid-Shunt Equivalent *M*-Derived Type.—A similar derivation is possible for the two structures in Figs. 8A and 8B. From equation (9) of Chapter XI, the mid-shunt iterative impedance of *each* of the structures shown in Figures 8A and 8B can be proved to be:

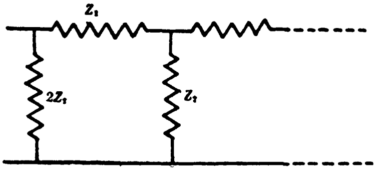

FIG. 8A. Prototype.

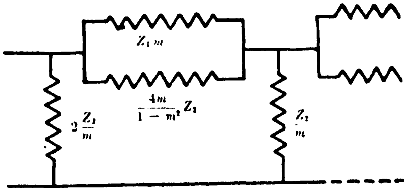

FIG. 8B. Mid-shunt equivalent m-derived type.

$$Z_K' = \sqrt{\frac{Z_1 Z_2}{1 + \frac{1}{4}\frac{Z_1}{Z_2}}} \tag{20}$$

Moreover, the ratio of the series to the shunt impedances in the case of Fig. 8B is, as in the case of Fig. 7B:

$$\left(\frac{Z_1}{Z_2}\right)_m = \frac{4m^2\left(\frac{Z_1}{Z_2}\right)}{(1 - m^2)\left(\frac{Z_1}{Z_2}\right) + 4} \tag{21}$$

It is, therefore, seen that there is a *second* derived type, whose mid-shunt iterative impedance is the same as that of the prototype from which it is derived, and consequently it is called the *mid-shunt equivalent m-derived type.* The value of $(Z_1/Z_2)_m$ and hence the value of the propagation constant of this type is, moreover, identically the same as that of the corresponding *mid-series equivalent m-derived type.*

It is apparent that the above method is a very important one since by its use it is only necessary to obtain direct formulae for the simpler types of filter structures. Formulae applying to the more complex structures may then be readily obtained by regarding these structures as derived types, as explained above. A specific application of this method will be used in determining the formulae for a three-element low pass wave filter.

16.4 Other Derived Types.

16.41 Equivalent Impedance (2-Terminal) Networks.—If one is given any combination of three (or more) elements and if the impedance of one of the elements varies in exactly the same way, with frequency, as does the impedance of a second element, then the three (or more) elements can, in general, be replaced by another combination of the same number and types of elements but having a different physical configuration and, in general, having different constants from those of the elements in the original combination.

For example, a structure consisting of an inductance, in series with a condenser and inductance in parallel, such as is shown in Fig. 9A, can always be replaced by an inductance, in parallel with a condenser and inductance connected in series, such as is shown in Fig. 9B. With proper relations (such as are given in formulae (36) to (41) of Appendix D) between the values of the inductances and the capacities in Figs. 9A and 9B, the impedance of either combination of elements will be identically the same at all frequencies from zero to infinity, and one combination of elements in any circuit can, therefore, always be replaced by the other combination without affecting in any way the action of the circuit. Specific formulae for the more commonly used combinations of inductances and condensers, which can be replaced, as explained above, by other equivalent impedance networks, are given in Sections II, III and IV of Appendix D as are also similar formulae for specific combinations of resistances and inductances or of resistances and condensers.

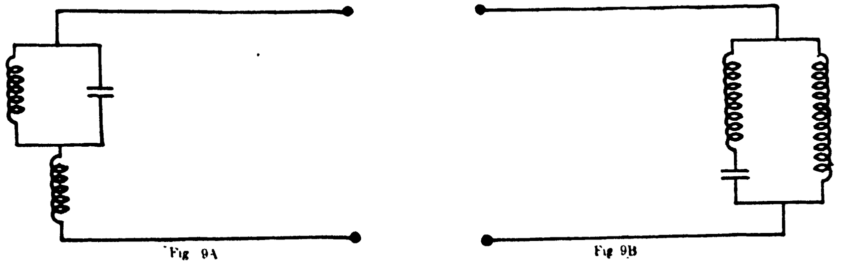

Fig 9A

Fig 9B

FIGS. 9A and 9B. Typical combinations of elements whose impedances can be made equal to each other *at all frequencies.*

The fact that any *T* network composed simply of three condensers (or inductances) could be replaced by a corresponding equivalent π network containing three condensers (or inductances), or vice versa, was pointed out by Kennelly 25 years ago. Relations for converting any such T network into a π network, or vice versa, were given by Kennelly,* and are shown in Figs. 27 and 28 of Appendix D.

16.42 Factors Determining the Choice of Any Particular Network. —It is evident that when one network can be made electrically equivalent to another network, the choice as to which one shall be used in any given case often depends upon manufacturing or commercial considerations. For example, consider the shunt arm of the mid-series equivalent *m-derived* type of band pass filter shown as filter No. 50 in Fig. 3. This arm consists of a condenser C_2 in series with an inductance L_2, in series with the parallel combination of a condenser C_3 and an inductance L_3. If the filter has a value of m that is near unity, L_2 becomes small as compared with L_1, and C_2 becomes large as compared with C_1. In other words, the values of the capacities and inductances vary over relatively large ranges.

Such relatively large variations in constants are extremely undesirable, especially in filters designed for high frequencies—such as filters for use in carrier systems. This is because small capacities (less than .001 mf.) are difficult to measure accurately, and because capacities in the wiring of the filters, etc., may often be several per cent of the total capacity of the condensers involved in the filter structure proper. Similarly, large values of capacity (greater than .3 mf.) are undesirable on account of the cost of such condensers, especially when made of mica—as is necessary at carrier current frequencies (5,000 to 30,000 cycles) in order to keep dissipation down to a negligible amount, and in order to keep variations in the values of the capacities, due to temperature changes, etc., within the required limits. Likewise, small values of inductances (less than .1 millihenry) are difficult to measure accurately, while large values of inductances (greater than 50 millihenries) are affected by the shunt capacities, between the windings, which make the effective inductances vary with the frequency.

In view of this, it is apparent that if the shunt arm of the filter referred to above be replaced by a structure in which the *relative* values of the inductances and of the capacities do not vary from each other by such large amounts, an improvement will be made in the design of the filter from a commercial standpoint. This statement is of course only true provided the capacities and inductances of the replacing structure are of equally suitable commercial values. Since all the inductances of any filter structure are always proportional *directly*, and all the capacities *inversely*, to its iterative impedance, the order of magnitude of the values of the inductances and capacities can usually be controlled satisfactorily by a proper choice of the iterative impedance of the filter. If the value of the terminal impedances between which the filter is to work is materially different from the most desirable value of the iterative impedance—from the standpoint of giving reasonable values for the inductances and condensers—transformers can usually be used at the ends of the filter to correct for this disparity in impedance.

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* "The Equivalence of Triangles and Three-Pointed Stars in Conducting Networks." *El. World and Engineer*, N. Y., Vol. XXXIV, No. 12, pp. 413–414, Sept. 16, 1899.

16.43 Examples of Derived Types.—The above discussion makes it evident that if the formulae are given for any filter structure, which has three or more elements in any one arm, at least one other structure can always be derived, by using an *equivalent impedance network*, which will have exactly the same characteristics as the original structure. The decision as to which mesh should be used in any given case is, then, usually a question of manufacturing expediency.

For example, referring to Fig. 3, it is seen that the mid-series equivalent *m-derived* type of filter (derived from filter No. 28 as a prototype), designated as filter No. 50, is the same as filters No. 49, No. 51 and No. 52 with the exception of its shunt arm. The conditions for the equivalence of the shunt arms, however, are given by formulae (62) to (81) of Appendix D. Hence, if the formulae applying to the original prototype, or filter No. 28, are known, the corresponding formulae for the *m-derived type* (filter No. 50) can immediately be derived—as previously shown—and from the *m-derived type* of filter one can, by aid of the equivalent network formulae given in Appendix D, immediately get the formulae applying to the filter structures designated as No. 49, No. 51 and No. 52.

Similarly, if formulae for the equivalent mid-shunt *m-derived* (from filter No. 28) type of filter designated as filter No. 54 in Fig. 3 are known, the corresponding formulae for the structures designated as filters No. 53, No. 55 and No. 56 can be immediately obtained.

From the foregoing, it is evident that if the formulae applying to any one of the filters noted below are known, the formulae for the other filter associated with it can immediately be derived from the equivalent network formulae. Filters shown in Fig. 3 which have three or four element networks that are equivalent to each other are:

Filter Nos. (Fig. 3)	Filter Nos. (Fig. 3)
4–5	37–38
6–7	39–40
11–12	41–42
13–14	43–44
16–17–18–19	45–46
20–21–22–23	47–48
31–32	49–50–51–52
33–34	53–54–55–56

The formulae connecting equivalent impedance or 2-terminal networks are very important since, in general, they make it possible to derive at once all the necessary formulae for any structure that employs a mesh equivalent to that of another structure, provided the formulae for the latter structure are known.

16.5 Lattice Structures.

16.51 Relation Between Lattice and Series-Shunt or Ladder Structures.—It has been pointed out* that any passive structure having two input and two output terminals may, at any given frequency, be reduced to an equivalent T or Π network. In like manner, from such relations as are shown in Fig. 24 of Appendix D, it is possible to show that any T or Π network, or any other structure of the ladder or series-shunt type which is composed of pure reactances, may be reduced to an equivalent structure of the lattice or bridge type—this latter structure being the equivalent of the former *at all frequencies*. The general formulae or relations by which such equivalence may be obtained are given in sketches 1 and 2 of Fig. 4. These relations make it possible to construct at once, from any T or Π network, an equivalent lattice network. Consequently, if the constants of any filter section of the series-shunt type are known, it is a very simple matter to determine the constants of the corresponding lattice type of filter section which will be the equivalent of the series-shunt or ladder type of structure.

16.52 Example of Derivation of Constants and Formulae.—Suppose, for example, that the formulae for—or the values of—L_0 and C_0 for the ladder type of structure listed as No. 4 in Fig. 4 are known and it is desired to get the values of the capacities C_1 and C_2 and the inductance L_1 of the equivalent lattice type of structure. From the relations given in sketch 2 of Fig. 4, it is at once apparent that:

$$Z_1 = jL_0\omega \quad \text{Whence} \quad \frac{1}{2}Z_1 = j\frac{L_0\omega}{2} = jL_1\omega \quad \text{or} \quad L_1 = L_2 = \frac{L_0}{2} \qquad (22)$$

$$2Z_2 = \frac{2}{jC_0\omega} \quad \text{But} \quad 2Z_2 = \frac{1}{jC_1\omega} = \frac{1}{jC_2\omega} \quad \text{Whence} \quad C_1 = C_2 = \frac{C_0}{2} \qquad (23)$$

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* See Chapter VIII.

It is seen, then, that the method outlined above furnishes a ready means of deriving all the formulae and constants for any *lattice type* of structure —provided the corresponding formulae and constants of the equivalent *ladder type* of structure are known.

16.6 Bridged-*T* Structures.—In a way that is analogous to that explained above for lattice structures, it is possible to derive the constants and formulae for a *bridged-T* structure provided the formulae and constants of the equivalent *ladder type* of structure are known. Typical examples of such equivalent structures, the formulae and constants for one of which may be thus readily derived when the formulae and constants of the other equivalent structure are known, are given in Fig. 5. As previously stated, the equivalence of most of these structures is usually self-evident when one replaces the *T* network of coils or condensers in one structure by its corresponding Π network of coils or condensers in the other structure.

CHAPTER XVII

DESIGN OF WAVE FILTERS

17.0 In order to show how the general formulae just obtained may be applied to specific types of structures, there will be developed all of the formulae ordinarily required for two typical structures, viz., two low pass filters—a simple type or prototype, and a derived type. An understanding of these typical derivations should make it possible for the reader to derive the corresponding formulae for other types of filters.

17.1 Low Pass Filters.

17.11 Two-Element Type.—The simplest type of low pass wave filter is that shown in Fig. 1. This filter has been called the *two-element type of low pass wave filter* since each periodic section contains only two different elements—one inductance and one condenser.

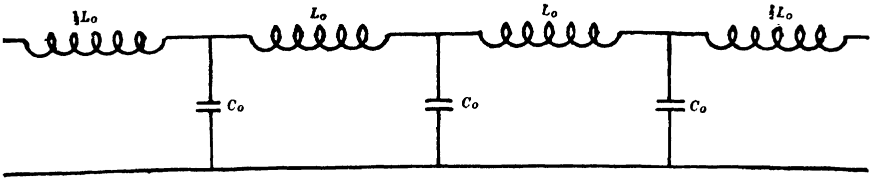

FIG. 1. Two-element type of low pass wave filter—no dissipation.

17.111 Cut-Off Frequency, f_c.—The impedance of the series arm is

$$Z_1 = jL_0\omega \tag{1}$$

and that of the shunt arm is

$$Z_2 = \frac{1}{jC_0\omega} \tag{2}$$

where $\omega \equiv 2\pi f$, f being the frequency. Whence, the ratio

$$\frac{Z_1}{Z_2} = -L_0C_0\omega^2. \tag{3}$$

The limiting frequencies which any non-dissipative filter will transmit have been shown to be (see Sect. 16.232) determined by

$$\frac{Z_1}{Z_2} = 0 \qquad \text{and} \qquad \frac{Z_1}{Z_2} = -4. \tag{4}$$

Hence, the above filter will transmit all frequencies lying between the limits given by

$$-L_0C_0\omega^2 = 0 \qquad \text{or} \qquad \omega = 0 \tag{5}$$

and

$$-L_0C_0\omega_c^2 = -4 \qquad \text{or} \qquad \omega_c = \frac{2}{\sqrt{L_0C_0}} \equiv 2\pi f_c \tag{6}$$

Consequently, the filter structure shown in Fig. 1 is a low pass filter transmitting currents of all frequencies lying between zero and the cut-off frequency, f_c, where

$$f_c = \frac{1}{\pi\sqrt{L_0C_0}} \tag{7}$$

17.112 Mid-Series Iterative Impedance.—The mid-series iterative impedance of any structure, which is designated as Z_K, has been shown to be (see formula (1), Chapter XI):

$$Z_K = \sqrt{Z_1Z_2}\sqrt{1 + \frac{1}{4}\frac{Z_1}{Z_2}} \tag{8}$$

Hence, for the structure shown in Fig. 1, this impedance is

$$Z_K = \sqrt{\frac{jL_0\omega}{jC_0\omega}}\sqrt{1 - \frac{L_0C_0\omega^2}{4}} = \sqrt{\frac{L_0}{C_0}}\sqrt{1 - \frac{L_0C_0\omega^2}{4}} \tag{9}$$

But from (6)

$$\frac{4}{L_0C_0} = \omega_c^2 = (2\pi f_c)^2 \tag{10}$$

Therefore

$$Z_K = \sqrt{\frac{L_0}{C_0}}\sqrt{1 - \left(\frac{\omega}{\omega_c}\right)^2} = \sqrt{\frac{L_0}{C_0}}\sqrt{1 - \left(\frac{f}{f_c}\right)^2} \tag{11}$$

It is to be noted from (8) and (9) that the product of Z_1 and Z_2 is independent of the frequency; that is, $\sqrt{Z_1Z_2} = \sqrt{L_0/C_0} \equiv K$. Such a structure is called a *constant-K structure* and it will be shown to have certain properties that are very useful—particularly in connection with the design of corrective networks. If Z_0 is defined as the *nominal iterative impedance of a constant-K structure, that is, the iterative impedance at the frequency at which* $Z_1/Z_2 = 0$, which in this case is at zero frequency, it is evident that

$$Z_0 = \sqrt{\frac{L_0}{C_0}} \tag{12}$$

and hence

$$Z_K = Z_0\sqrt{1 - \left(\frac{f}{f_c}\right)^2} \tag{13}$$

17.113 Mid-Shunt Iterative Impedance.—The mid-shunt iterative impedance of any structure, which is designated as Z_K', has been shown to be (see formula (9), Chapter XI):

$$Z_K' = \frac{\sqrt{Z_1 Z_2}}{\sqrt{1 + \frac{1}{4}\frac{Z_1}{Z_2}}} \tag{14}$$

Hence, from what has been seen in Sect. 17.112, it is evident that for the structure under consideration the mid-shunt iterative impedance is:

$$Z_K' = \frac{Z_0}{\sqrt{1 - \left(\frac{f}{f_c}\right)^2}} \tag{15}$$

17.114 Design Formulae.—In the design of a low pass filter structure the available information usually includes (1) the cut-off frequency f_c and (2) the terminal impedances between which the filter is to operate. If these impedances are, as is usually the case, pure resistances, Z_R, and if they are the same at both ends of the filter, it is easy to prove, as pointed out in Sect. 11.8, that the transmission loss caused by the filter in the unattenuated range will be a minimum (i.e., it will be simply the loss due to the attenuation of the filter) provided the iterative impedance of the filter has the same value as do each of the terminal impedances.

A plot of equation (13) is given by the $a = \infty$ (infinity) curve in Fig. 2 and shows that for frequencies not too close to the cut-off frequency the iterative impedance of the filter remains reasonably close to that of Z_0 or the nominal iterative impedance. Hence, it is generally assumed that the Z_0 of a filter should be of approximately the same order of magnitude as the terminal impedances between which the filter is to be connected. An additional justification for this assumption will be seen when consideration is given to the iterative impedance curves of derived types of structures—such as are shown by the other curves in Fig. 2.

If the cut-off frequency f_c and the nominal iterative impedance Z_0 are known, formulae for the values of the series inductance L_0 and the shunt capacity C_0 can be obtained by solving equations (10) and (12) simultaneously. This solution gives

$$L_0 = \frac{Z_0}{\pi f_c} \tag{16}$$

and

$$C_0 = \frac{1}{\pi f_c Z_0} \tag{17}$$

17.115 Propagation Constant P of Dissipative Filter.—In any actual filter there is, of course, a certain amount of resistance and energy dissipa-

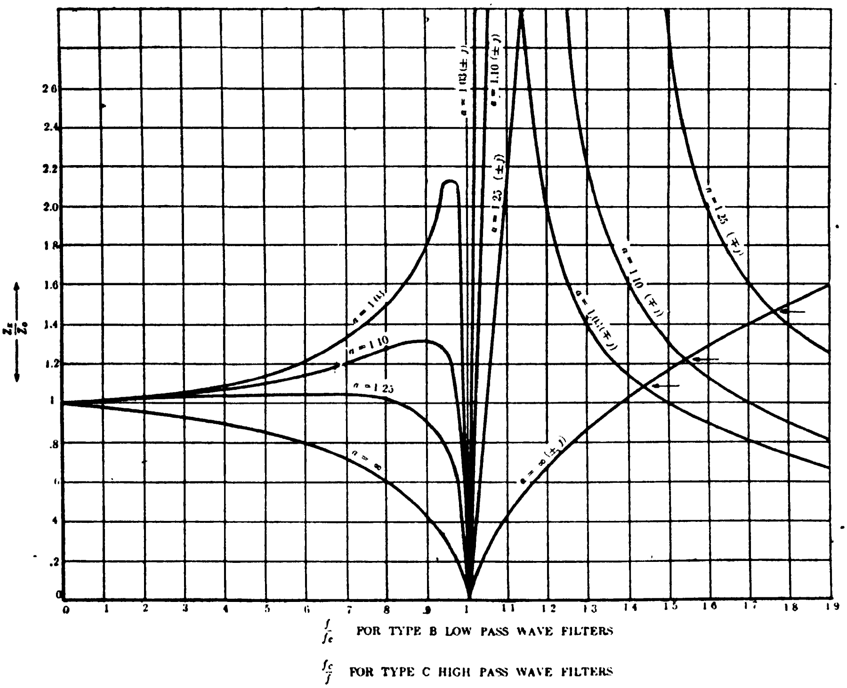

FIG. 2. Curves showing the ratio of the iterative impedance (mid-series) at any frequency to that at zero frequency, if a ("Type B") low pass filter, or to that at infinite frequency, if a ("Type C") high pass filter.

tion associated with the inductances and condensers. At ordinary voice frequencies, the dissipation in both paper and mica condensers is usually negligible. Similarly, at carrier frequencies, say from 3,000 cycles to 100,000 cycles, where only mica condensers are ordinarily used, the dissipation in the condensers can likewise be neglected. However, the effects of dissipation in the coils of a filter are seldom negligible and in general must be considered.

Consider, then, the same structure as previously discussed except that each coil has an effective resistance R which, it will be assumed, cannot be neglected. Such a structure is indicated in Fig. 3. It has already

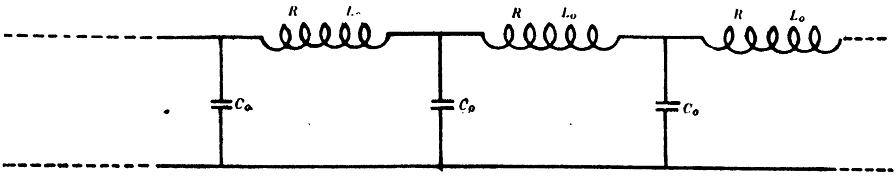

FIG. 3. Two-element type of low pass wave filter—having dissipation.

been shown that the propagation constant P of any structure is dependent simply on the ratio of the series to the shunt arms of the structure; that is, upon Z_1/Z_2. *In other words, it is possible to read directly, from curves, such as are given in Figs. 11 and 12 of Appendix E, the attenuation constant* (A) *or the phase constant* (B) *per periodic section of any ladder type of structure when the value of* $Z_1/4Z_2$ *is known.*

Considering, then, Fig. 3, it is evident that

$$Z_1 = R + jL_0\omega \qquad \text{and} \qquad Z_2 = \frac{1}{jC_0\omega} \tag{18}$$

If Q is the ratio of the reactance of the coil to its effective resistance—this ratio being called the *coil dissipation constant*—or if

$$Q \equiv \frac{L_0\omega}{R} \tag{19}$$

then

$$Z_1 = L_0\omega\left(\frac{1}{Q} + j1\right) \tag{20}$$

and

$$\frac{Z_1}{Z_2} = L_0\omega\left(\frac{1}{Q} + j1\right) \times jC_0\omega = \frac{-L_0C_0\omega^2(Q - j1)}{Q} \tag{21}$$

But from (6)

$$L_0C_0 = \frac{4}{\omega_c^2} \tag{22}$$

Whence

$$\frac{Z_1}{Z_2} = -4\left(\frac{f}{f_c}\right)^2 \times \frac{Q - j1}{Q} \tag{23}$$

In the ideal (non-dissipative) case Q is, of course, infinite and equation (23) becomes

$$\frac{Z_1}{Z_2} = -4\left(\frac{f}{f_c}\right)^2 \tag{24}$$

This completes the derivation of the more important formulae applying to the two-element low pass type of filter.

17.12 Mid-Series Equivalent *M*-Derived Type.—Considering the two-element filter just discussed as the prototype, the *mid-series equivalent m-derived type* will be (see Fig. 7B of Chap. XVI) a structure of the form and having the constants shown in Fig. 4B. This sort of structure has been called a *three-element* type of low pass wave filter.

17.121 Mid-Series Iterative Impedance Z_K of a Non-Dissipative Filter.—Since it has been shown in Sect. 16.31 that the above structures

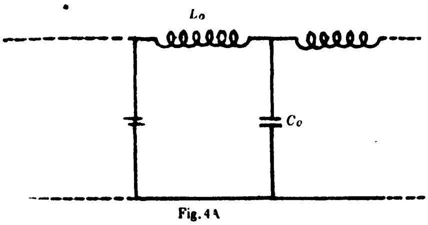

FIG. 4A. Prototype of low pass filter.

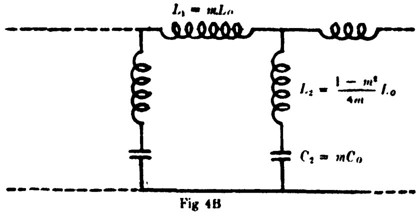

FIG. 4B. Mid-series equivalent *m*-derived type of low pass filter.

have the same mid-series iterative impedances and the same cut-off frequencies, it is evident from (11) that the mid-series iterative impedance of the three-element non-dissipative low pass filter shown in Fig. 4B is

$$Z_K = \sqrt{\frac{L_0}{C_0}\left[1 - \left(\frac{f}{f_c}\right)^2\right]} \tag{25}$$

But in the derived type, as shown in Fig. 4B, $L_1 = mL_0$ and $C_2 = mC_0$. Hence

$$\sqrt{\frac{L_1}{C_2}} = \sqrt{\frac{mL_0}{mC_0}} = \sqrt{\frac{L_0}{C_0}} \tag{26}$$

Whence from (25) the mid-series iterative impedance of the three-element structure is

$$Z_K = \sqrt{\frac{L_1}{C_2}}\left[\sqrt{1 - \left(\frac{f}{f_c}\right)^2}\right] \tag{27}$$

From equations (12) and (26),

$$Z_0 = \sqrt{\frac{L_1}{C_2}} \tag{28}$$

Whence

$$Z_K = Z_0\sqrt{1 - \left(\frac{f}{f_c}\right)^2} \tag{29}$$

which, as previously stated, is the same as that obtained for the two-element filter having the same nominal iterative impedance, Z_0.

17.122 Cut-Off Frequency, f_c.—From (7) it is clear that in the prototype, the cut-off frequency f_c is

$$f_c = \frac{1}{\pi\sqrt{L_0C_0}} \tag{30}$$

which must also be equal to the cut-off frequency of the derived type as shown in Sect. 16.31. In the derived structure (see Fig. 4B)

$$L_1 = mL_0 \tag{31}$$

$$L_2 = \frac{1 - m^2}{4m} L_0 \tag{32}$$

and

$$C_2 = mC_0 \tag{33}$$

Whence

$$C_0 = \frac{C_2}{m} \quad \text{and} \quad m = \frac{L_1}{L_0} \tag{34}$$

or

$$C_0 = C_2 \times \frac{L_0}{L_1} \tag{35}$$

Also from (31) and (32)

$$L_2 = \frac{1 - \left(\frac{L_1}{L_0}\right)^2}{\frac{4L_1}{L_0^2}} = \frac{L_0^2 - L_1^2}{4L_1} \tag{36}$$

or

$$L_0^2 = L_1(L_1 + 4L_2) \tag{37}$$

The cut-off frequency of the derived type of filter is thus

$$f_c = \frac{1}{\pi\sqrt{L_0C_0}} = \frac{1}{\pi\sqrt{L_0 \times C_2 \times \frac{L_0}{L_1}}} = \frac{1}{\pi\sqrt{\frac{C_2}{L_1} \times L_0^2}} \tag{38}$$

or

$$f_c = \frac{1}{\pi\sqrt{\frac{C_2}{L_1} \times L_1(L_1 + 4L_2)}} = \frac{1}{\pi\sqrt{C_2(L_1 + 4L_2)}} \tag{39}$$

17.123 Frequency of Infinite Attenuation, f_∞.—Infinite attenuation in non-dissipative filters will occur in the derived or m-type when $(Z_1/Z_2)_m$ is infinite. It is evident from equation (19) of Chap. XVI that this occurs when

$$\frac{Z_1}{Z_2} = \frac{-4}{1 - m^2} \tag{40}$$

But from (31) and (32) it is known that

$$m^2 = \frac{1}{1 + 4\frac{L_2}{L_1}} \quad \text{or} \quad m = \frac{1}{\sqrt{1 + 4\frac{L_2}{L_1}}} \tag{41}$$

and from (24) that

$$\frac{Z_1}{Z_2} = -4\left(\frac{f}{f_c}\right)^2 \tag{42}$$

Hence, the frequency f_∞ of infinite attenuation is given by

$$\left[-4\left(\frac{f_\infty}{f_c}\right)^2\right] = \frac{-4}{1 - \dfrac{1}{1 + 4\dfrac{L_2}{L_1}}} \tag{43}$$

or *the ratio of the frequency of infinite attenuation to the cut-off frequency which, in low pass filters, is usually called "a"* is:

$$a \equiv \frac{f_\infty}{f_c} = \sqrt{1 + \frac{1}{4}\frac{L_1}{L_2}} \tag{44}$$

Since it has been shown in (39) that

$$f_c = \frac{1}{\pi\sqrt{C_2(L_1 + 4L_2)}} \tag{45}$$

then

$$f_\infty = \frac{1}{2\pi\sqrt{L_2C_2}} \tag{46}$$

In passing, it may be remarked that equations (41) and (44) give a relation which is often useful, namely:

$$m = \frac{\sqrt{a^2 - 1}}{a} \qquad \text{or} \qquad a = \frac{1}{\sqrt{1 - m^2}} \tag{47}$$

17.124 Determination of the Values of the Filter Elements.—From equations (31), (47) and (16)

$$L_1 = L_0m = L_0\frac{\sqrt{a^2 - 1}}{a} = \frac{Z_0}{\pi f_c}m = \frac{Z_0}{\pi f_c}\frac{\sqrt{a^2 - 1}}{a} \tag{48}$$

Similarly, from equations (32), (47) and (16)

$$\begin{aligned} L_2 = L_0\frac{1 - m^2}{4m} = L_0\frac{1}{4a\sqrt{a^2 - 1}} &= \frac{Z_0}{\pi f_c} \times \frac{1 - m^2}{4m} \\ &= \frac{Z_0}{\pi f_c} \times \frac{1}{4a\sqrt{a^2 - 1}} \end{aligned} \tag{49}$$

and from equations (33), (47) and (17)

$$C_2 = C_0m = C_0\frac{\sqrt{a^2 - 1}}{a} = \frac{1}{\pi f_cZ_0}m = \frac{1}{\pi f_cZ_0}\frac{\sqrt{a^2 - 1}}{a} \tag{50}$$

These formulae enable one to determine all the constants of the filter when the nominal iterative impedance Z_0, the cut-off frequency f_c and the ratio f_∞/f_c (or a) are known. It is to be noted that if the variable $a = \infty$, the value of L_2 becomes zero and the 3-element or the mid-series equivalent *m-derived* type of low pass filter reduces to the 2-element type or to its prototype.

17.125 Ratio of the Series to the Shunt Impedances, Z_1/Z_2.—As previously explained, this ratio completely determines, for any structure, the propagation constant per periodic interval of the structure.

Referring to Fig. 4B

$$Z_1 = R_1 + jL_1\omega = \frac{L_1\omega}{Q} + jL_1\omega \tag{51}$$

and

$$Z_2 = R_2 + j\left(L_2\omega - \frac{1}{C_2\omega}\right) = \frac{L_2\omega}{Q} + j\frac{L_2C_2\omega^2 - 1}{C_2\omega} \tag{52}$$

Whence

$$\frac{Z_1}{Z_2} = \frac{(1 + jQ)(L_1C_2\omega^2)}{L_2C_2\omega^2 + jQ(L_2C_2\omega^2 - 1)} \tag{53}$$

Putting the values of L_1, L_2 and C_2, as given by equations (48), (49) and (50), in the above equation,

$$\frac{Z_1}{Z_2} \equiv U + jV = \frac{4(1 + jQ)(a^2 - 1)}{1 + jQ\left[1 - \frac{a^2f_c^2}{f^2}\right]} \tag{54}$$

In the ideal case where there is no dissipation; i.e., where Q is infinite,

$$\frac{Z_1}{Z_2} \equiv U + jV = \frac{4(a^2 - 1)}{1 - a^2\left(\frac{f_c}{f}\right)^2} \tag{55}$$

These last two formulae are, as will be shown later, extremely useful in connection with the determination of the propagation constant and, hence, are important when calculating the transmission loss occurring in a dissipative low pass filter.

17.126 Mid-Shunt Iterative Impedance Z_K' of a Non-Dissipative Filter.—From equations (1) and (9) in Chapter XI it is seen that the mid-shunt iterative impedance Z_K' of any structure is

$$Z_K' = \frac{Z_K}{1 + \frac{1}{4}\frac{Z_1}{Z_2}} \tag{56}$$

With the aid of equation (55) the above equation gives

$$Z_K' = \frac{Z_K}{1 + \dfrac{a^2 - 1}{1 - a^2\left(\dfrac{f_c}{f}\right)^2}} = Z_K \frac{1 - \left(\dfrac{f}{af_c}\right)^2}{1 - \left(\dfrac{f}{f_c}\right)^2} \qquad (57)$$

But it has been seen from (29) that

$$Z_K = Z_0\sqrt{1 - \left(\frac{f}{f_c}\right)^2}$$

Hence

$$Z_K' = \frac{Z_0\left[1 - \left(\dfrac{f}{af_c}\right)^2\right]}{\sqrt{1 - \left(\dfrac{f}{f_c}\right)^2}} \qquad (58)$$

It should be noted, from (58), that the value of Z_K' is very nearly constant over the entire transmitting range of the filter and is essentially equal to Z_0—provided a has a value of approximately 1.25. This may be seen by reference to the curves of Fig. 2 which, although as indicated, apply to the mid-series iterative impedance of an equivalent mid-shunt *m-derived* type of low pass wave filter, are also, in effect, a plot of Z_0/Z_K'. Such terminations evidently make it possible to eliminate practically all reflection effects over the greater portion of the transmitting range of a filter.

This completes the derivation of the more important formulae applying to the mid-series equivalent derived type of low pass filter shown in Fig. 4B.

17.2 Propagation Characteristics of Non-Dissipative Filters.

17.21 Attenuation Characteristics.—As pointed out in Sect. 17.42, the attenuation constant of any properly designed composite wave filter may be obtained by adding together the attenuation constants of the various sections of which the structure is composed. The attenuation characteristics of various types of wave filter sections are shown, in a symbolic way, in Fig. 3 of Chapter XVI. There are also shown in Fig. 5 the attenuation characteristics of low pass and high pass wave filter sections having various values of a and a'. The meaning of a' is given in Sect. 17.212.

17.211 Structures Having Infinite Attenuation at Some Frequency.—It is seen from equation (55) that if a has a finite value, Z_1/Z_2

becomes a minimum and, hence, the value of the attenuation constant of a 3-element low pass wave filter becomes a minimum at an infinitely high frequency. A quantitative idea of this effect may be gained by comparing the attenuation curves, in Fig. 5, of a wave filter section whose $a = 1.01$ with that of a section whose $a = 1.50$. Fig. 5 also indicates clearly the advantages which may be obtained, from an attenuation standpoint, of using a composite wave filter structure or, more specifically, a structure composed of one or more sections of the 2-element ($a = \infty$) type—which give only a relatively small attenuation near the cut-off frequency but a very large attenuation at frequencies remote from the cut-off frequency—combined with one or more sections of the *m-derived* type, having different values of m or a. Such combinations will evidently give a structure whose attenuation characteristics are steep near the cut-off frequency and are maintained at relatively high values at frequencies remote from the cut-off frequency. The method of joining together such sections without reflection effects is discussed in Sect. 17.4.

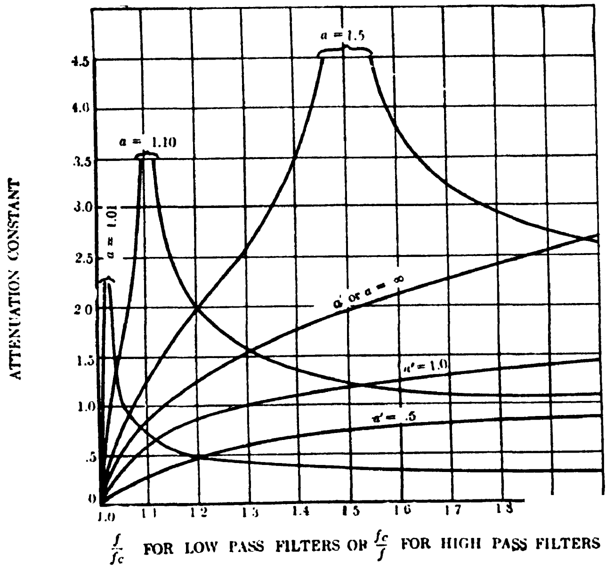

FIG. 5. Attenuation constant—frequency ratio curves for various types of low pass and high pass wave filters.

17.212 Structures Never Having Infinite Attenuation.—There are certain types of low, high and band pass wave filter sections that never have infinite attenuation at *any* frequency. All such wave filter sections employ coils having mutual inductance between two or more of their windings. Referring to Fig. 5, the attenuation characteristics of these structures are seen to approach that of the 2-element structure when $a' = \infty$, and to approach zero at all frequencies when $a' = 0$. In the case of a L.P. filter, a' is defined as f_a/f_c, and in the case of a H.P. filter, as f_c/f_a—where f_a is a frequency more or less arbitrarily chosen.

17.22 Phase Shift Characteristics.—The imaginary component of the image transfer constant, which has variously been called the *phase shift constant*, the *wave-length constant* or the *phase constant*, is not ordinarily of as much interest as is the attenuation constant of a wave filter structure. This is because—as Von Helmholtz pointed out—the ear is unable, in ordinary acoustic systems, to detect phase relations. In other words, it is largely the *absolute magnitude* of the received currents at the various frequencies that determines the sensations produced upon the auditory nerves. However, phase relations have, indirectly, some effect upon the absolute magnitude of the received current in a telephone system, as may be seen from a consideration of the interaction factor or the last factor in equation (54) of Chapter XI. Ordinarily, however, it is only in the transmitting range of a filter and only near one of its cut-off frequencies that the interaction factor and, hence, the phase constant becomes important in the ordinary telephone system. Another place where the phase constant-frequency characteristic is of importance is in connection with the use of low pass filter structures, or their equivalents, in binaural systems. In such systems—which are likely to gain in importance as time goes on—it is usually desirable to employ a structure in which the phase constant is as nearly as possible directly proportional to the frequency. Although probably of sufficient importance to warrant a more complete analysis, the phase constant is here only briefly discussed because of limited space.

17.221 Structures Having Infinite Attenuation at Some Frequency.—In structures having infinite attenuation at some frequency, the phase constant per section is ordinarily determined most readily from the value of $Z_1/4Z_2$ of the equivalent ladder type of structure and from curves of the general type illustrated in Fig. 12 of Appendix E. It can readily be shown that the amount of dissipation usually encountered in a wave filter has ordinarily but little effect upon the phase constant of the structure. Hence, it is a sufficiently good approximation in most cases to obtain the phase constant from curves or computations which are calculated on a non-dissipative basis. Fig. 6 shows a set of such curves for various values of a. From these curves, which apply only to low and high pass wave filter sections, it may be seen that there is always a shift of π radians or 180° at the cut-off frequency. The phase shift at any lower frequency, in a low pass filter, and at any higher frequency, in a high pass filter, is dependent upon the value of *a*. In composite wave filter structures the total phase constant is equal to the sum of the phase constants of the individual sections provided all adjoining sections have the same image impedances at their junctions.

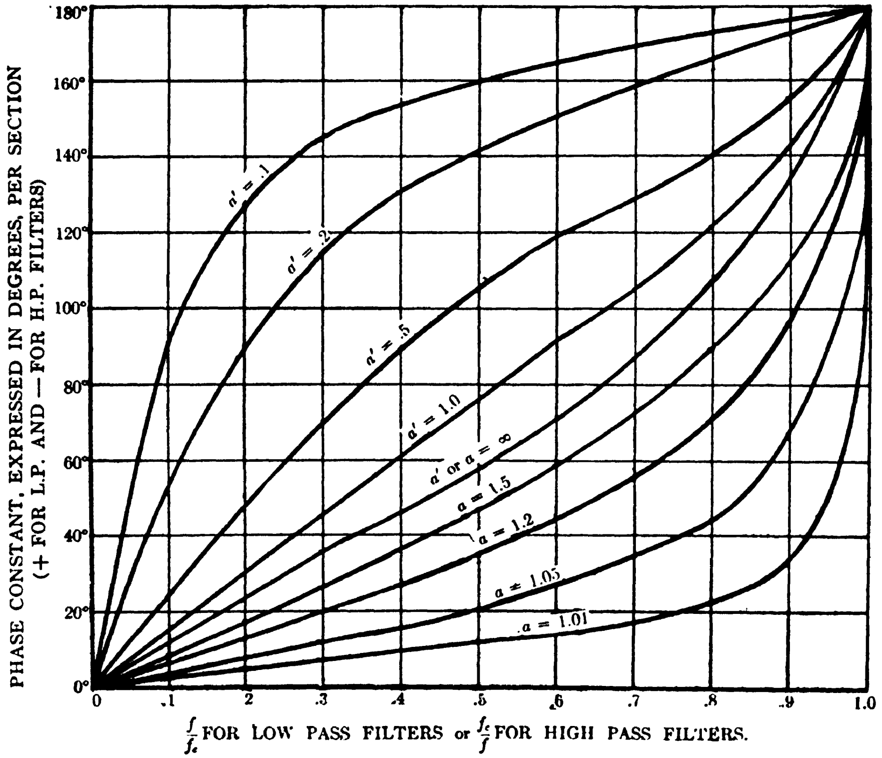

Fig. 6. Phase constant—frequency ratio curves for various types for low pass and high pass wave filters.

17.222 Structures Never Having Infinite Attenuation.—The phase shift in low pass and high pass wave filter sections, of the type that never has infinite attenuation, may also be obtained from curves such as are shown in Fig. 6. Here again it is to be noted that the phase constant or the phase shift at the cut-off frequency is always π radians or 180° per section. The phase shift at lower frequencies, in low pass filters, and at higher frequencies, in high pass wave filters, is always less than 180° and is a function of a' or of the specific make-up of the wave filter section.

In the case of binaural systems where, as previously stated, it is frequently desired to have the phase shift approximately proportional to the frequency, it is seen that the value of a' should be usually in the neighborhood of unity. The exact value of a' which should be chosen depends, however, upon (1) how close to the cut-off frequency it is desired to go and (2) what percentage deviation from linearity may be permitted.

17.3 Half Wave Filter Sections.—Up to the present point in the discussion of wave filter sections we have assumed the structure under consideration to be composed of an infinite number of recurrent *symmetrical* sections. Moreover, the formulae for the iterative impedances and the propagation constant have been derived on the same assumptions, namely, that of an infinitely recurrent structure. As already shown in Chapter XI, exactly similar results would have been obtained if any filter section had been considered as a unit and if formulae for the image impedances, image transfer constants, etc., had been derived on such a basis.

In the actual design of wave filters, as referred to in Sect. 17.62, it is frequently found desirable to make use of *half sections rather than full sections*. Whereas a full wave filter section is a symmetrical structure and may be replaced at any given frequency by a symmetrical T network, a half section (as shown in Fig. 7) will be a dissymmetrical structure and can be represented by a dissymmetrical T network in which the impedance of one of the series arms is zero—i.e. by an L network.

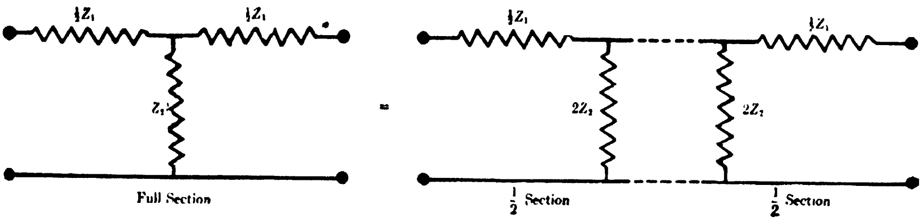

FIG. 7. Diagrams showing how a symmetrical T network can be regarded as comprised of two dissymmetrical or half sections.

It has been mentioned before, but not proven, that it is more convenient, when considering a *dissymmetrical* structure, to deal with its image impedances and image transfer constant, rather than with its iterative impedances and propagation constant. The distinct advantage of using the former, in the actual design of wave filters, will be made clear in the next few sections.

17.31 Image Impedances of a Half Section.—Consider Fig. 8 which represents one of the half sections shown in Fig. 7. Since the image impedances of any structure are determined by the short circuit and the open circuit impedances (see equations (25) and (26) of Chapter XI) of the structure

$$Z_{I_1} = \sqrt{\tfrac{1}{2}Z_1(\tfrac{1}{2}Z_1 + 2Z_2)} = \sqrt{Z_1Z_2\left(1+\frac{1}{4}\frac{Z_1}{Z_2}\right)} \tag{59}$$

and

$$Z_{I_2} = \sqrt{2Z_2\left(\frac{Z_1Z_2}{\frac{1}{2}Z_1 + 2Z_2}\right)} = \sqrt{\frac{Z_1Z_2}{1+\frac{1}{4}\frac{Z_1}{Z_2}}} \tag{60}$$

Referring to equations (1) and (9) of Chapter XI, it will be noticed that equations (59) and (60) are the mid-series and mid-shunt iterative impedances of the corresponding full section shown in Fig. 7. The image impedances Z_{I_1} and Z_{I_2}, then, of any half section, such as is shown in Fig. 8, can be obtained immediately from the full section impedances by the following relation: *The image impedances, Z_{I_1} and Z_{I_2}, of any half section are equal respectively to the mid-series and mid-shunt iterative impedances of the corresponding full section.*

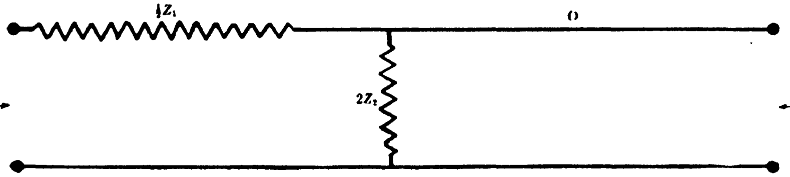

FIG. 8. Half section of a ladder type of structure.

It can be easily shown that, if the iterative impedances of the half section had been employed instead of the image impedances, no such simple relationship between the impedances of a full and a half section could have been obtained.

17.32 Image Transfer Constant of a Half Section.—Referring again to Fig. 8, the expression for the image transfer constant Θ of a half section is (see equation 94 of Chapter VII)

$$\Theta = \tanh^{-1}\sqrt{\frac{\frac{1}{2}Z_1}{\frac{1}{2}Z_1 + 2Z_2}} = \tanh^{-1}\sqrt{\frac{Z_1}{Z_1+4Z_2}} = \sinh^{-1}\tfrac{1}{2}\sqrt{\frac{Z_1}{Z_2}} \tag{61}$$

From equation (1) of Chapter XVI, (61) is seen to be equal to one-half of the propagation constant or of the image transfer constant of the corresponding full section. *The image transfer constant, therefore, of any half section is equal to one-half the image transfer constant (or to one-half of the propagation constant) of the corresponding full section.*

Again it can be easily shown that no such simple relationship exists between the propagation constant of a half section and that of the corresponding full section.

17.4 Composite Wave Filters.—In actual practice, a filter is usually composed of a number of sections (the number ranging from one-half a section to, say, six sections) in which all of the sections are so designed that they have the same image impedances at their respective junctions and in which the respective sections generally have different attenuation characteristics. Such a filter structure is called a *composite wave filter.* The desirability and advantages of using a number of dissimilar sections have been pointed out in Sect. 17.211.

17.41 Image Impedances, Z_{I_1} and Z_{I_2}, of a Composite Wave Filter.—In general, a composite wave filter may be regarded as a dissymmetrical structure; that is, it may be replaced at any given frequency, in so far as its external action is concerned, by a dissymmetrical T network which will have different image impedances at its two ends. A simple method will now be given for obtaining the image impedances of a composite filter of any number of sections, provided the sections are so designed that they have like image impedances at their respective junctions.

Referring to Fig. 9, consider two dissymmetrical T networks, *whose image impedances at their junction are equal.* From equations (25) and (26) of Chapter XI

$$Z_{I_a} = \sqrt{\frac{(Z_a + Z_c)(Z_aZ_b + Z_aZ_c + Z_bZ_c)}{Z_b + Z_c}} \tag{62}$$

$$\begin{aligned} Z_{I_b} &= \sqrt{\frac{(Z_b + Z_c)(Z_aZ_b + Z_aZ_c + Z_bZ_c)}{Z_a + Z_c}} \\ &= Z_{I_d} = \sqrt{\frac{(Z_d + Z_f)(Z_dZ_e + Z_dZ_f + Z_eZ_f)}{Z_e + Z_f}} \end{aligned} \tag{63}$$

$$Z_{I_e} = \sqrt{\frac{(Z_e + Z_f)(Z_dZ_e + Z_dZ_f + Z_eZ_f)}{Z_d + Z_f}} \tag{64}$$

where the image impedances of one T network are Z_{I_a} and Z_{I_b}, and those of the other are Z_{I_d} and Z_{I_e}.

From Figs. 2A and 2B of Chapter VIII it is possible to represent the T networks of Fig. 9 by a single equivalent T network whose arms will

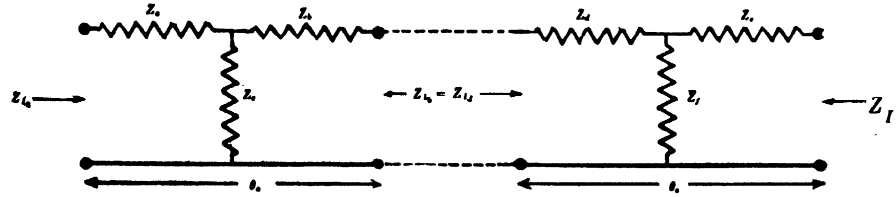

FIG. 9. Two dissymmetrical T networks having the same image impedances at their junction.

have the values shown in Fig. 10. The image impedance Z_{I_1} of this equivalent T network, as determined from the left end, is

$$Z_{I_1} = \sqrt{\frac{(Z_1 + Z_3)(Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3)}{Z_2 + Z_3}} \tag{65}$$

Substituting in (65) the values of Z_1, Z_2 and Z_3 as given in Fig. 10 and

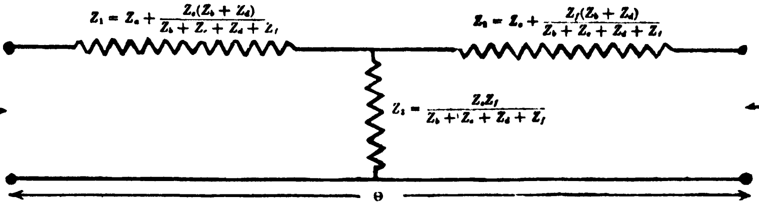
$Z_1 = Z_a + \frac{Z_c(Z_b + Z_d)}{Z_b + Z_c + Z_d + Z_f}$
$Z_2 = Z_e + \frac{Z_f(Z_b + Z_d)}{Z_b + Z_c + Z_d + Z_f}$
Z_{I_1}
$Z_3 = \frac{Z_c Z_f}{Z_b + Z_c + Z_d + Z_f}$
Z_{I_2}
θ

FIG. 10. Single T network equivalent to the two networks of Fig. 9.

making use of the relations expressed in (63), Z_{I_1} may be shown to be

$$Z_{I_1} = \sqrt{\frac{(Z_a + Z_c)(Z_a Z_b + Z_a Z_c + Z_b Z_c)}{Z_b + Z_c}} = Z_{I_a} \tag{66}$$

Similarly, it may be shown that

$$Z_{I_2} = \sqrt{\frac{(Z_e + Z_f)(Z_d Z_e + Z_d Z_f + Z_e Z_f)}{Z_d + Z_f}} = Z_I \tag{67}$$

It is evident from the above general proof that, in this manner, any number of symmetrical or dissymmetrical T networks may be combined with the result that the image impedances Z_{I_1} and Z_{I_2} of the T network—which is the equivalent of the entire structure—will be equal respectively to the terminating image impedances of the first and last T networks. The above result can be applied to the case of a composite wave filter by considering any full section of the filter to be represented by an equivalent symmetrical T network and any half section by an equivalent dissymmetrical T network in which the impedance of one of the series arms is zero. The above result may then be expressed as follows: *The image impedances, Z_{I_1} and Z_{I_2}, of a composite structure comprising any number of full or half sections are equal respectively to the terminating image impedances of the first and last half or full sections, provided that all the sections of the filter are so designed that they have equal image impedances at their respective junctions.*

17.42 Image Transfer Constant, Θ, of a Composite Wave Filter.—By following a method of proof similar to that used in the preceding section, we can arrive at a convenient and simple way of obtaining the total image transfer constant of a composite wave filter—by means of a knowledge of the image transfer constants of the various sections. Referring to Fig. 9, and to equation (27) of Chapter XI, it will be remembered that:

$$\Theta_a = \tanh^{-1}\sqrt{\frac{Z_aZ_b + Z_aZ_c + Z_bZ_c}{(Z_a + Z_c)(Z_b + Z_c)}} \equiv \tanh^{-1} x = \log_e\sqrt{\frac{1+x}{1-x}} \quad (68)$$

and

$$\Theta_e = \tanh^{-1}\sqrt{\frac{Z_dZ_e + Z_dZ_f + Z_eZ_f}{(Z_d + Z_f)(Z_e + Z_f)}} \equiv \tanh^{-1} y = \log_e\sqrt{\frac{1+y}{1-y}} \quad (69)$$

Adding (68) and (69):

$$\Theta_a + \Theta_e = \log_e\sqrt{\frac{1+x}{1-x}} + \log_e\sqrt{\frac{1+y}{1-y}} = \log_e\sqrt{\frac{(1+x)(1+y)}{(1-x)(1-y)}} \quad (70)$$

Now, referring to Fig. 10, the image transfer constant of the *T* network, which is the equivalent of the two *T* networks of Fig. 9, is

$$\Theta = \tanh^{-1}\sqrt{\frac{Z_1Z_2 + Z_1Z_3 + Z_2Z_3}{(Z_1 + Z_3)(Z_2 + Z_3)}} \equiv \tanh^{-1} z = \log_e\sqrt{\frac{1+z}{1-z}} \quad (71)$$

Substituting in (71) the values of Z_1, Z_2 and Z_3 as given in Fig. 10 and making use of the relations expressed in (63), equation (71) may be shown to be equal to equation (70), or in other words

$$\Theta = \Theta_a + \Theta_e \quad (72)$$

As in the preceding section, this general proof may be applied to the case of a composite filter to show that *the total image transfer constant,* Θ, *of a composite structure comprised of any number of full or half sections is equal to the sum of the image transfer constants,* $\Theta_1 + \Theta_2 + \cdots$, *etc., of the various full and half sections, provided that all the sections of the structure are so designed that they have equal image impedances at their respective junctions.*

17.5 Graphical Method of Determining Reflection and Terminal Losses.*—It was shown in Sect. 11.8 that the total transmission loss in any structure can, in many cases, be closely approximated by simply adding up the attenuation losses in the filter and the losses corresponding to the reflection effects at the terminals of the filter. Such a method is usually much simpler than the rigorous mesh method discussed in Sect. 9.0 due to the fact that the latter method simultaneously carries with it the determination of the total phase change as well as the determination of the absolute ratio of the currents. Since ordinarily a knowledge of the phase change is not required, the mesh method does more than is necessary. Again, with the mesh method, an alteration in the wave-filter design, as is sometimes desired, may require a complete recalculation —a thing that is seldom necessary when the other method is used. The mesh method is, of course, the only one adapted for use with general networks, but the much shorter method of adding the attenuation and reflection losses is practically always possible in wave filter structures as ordinarily designed. This is due to the fact that the image impedances of the adjacent sections comprising the filter structure are practically always made identical to each other, thus eliminating any reflection effects in the body of the structure. The attenuation losses are easily obtained from a knowledge of the values of the ratio, Z_1/Z_2, for the various component sections, and from curves (see Fig. 11, Appendix E) plotted from equation (7) of Chapter XI. The problem then is to determine what can be done in the way of plotting general curves to reduce the labor of calculating the values of the reflection or terminal losses.

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* This method was developed by O. J. Zobel, and is discussed by him, in detail, in a paper entitled "Transmission Characteristics of Electric Wave-Filters," *Bell System Technical Journal*, October, 1924.

Consider that it is desired to calculate, in accordance with the general method discussed in Sect. 11.8, the transmission loss of a filter on the assumption that the interaction factor can be neglected. As pointed out in the above mentioned section, in such cases it is only necessary to add to the attenuation losses of the various sections the values of the reflection factors at each end of the filter. The total attenuation losses of the various whole and half sections comprising the filter can be obtained by the method already discussed—when the values of the ratio Z_1/Z_2 are known for the various sections. In getting the value of the losses due to reflection effects, it is, as has been stated above, desirable to obtain general curves showing the values of reflection losses for various terminating conditions. Formulae for plotting such curves are derived in the following sections.

17.51 Filters Terminating in Mid-Section *M*-Derived Structures. —The first of these general curves is based on the fact that most filters are terminated either in a mid-shunt termination of a mid-series equivalent *m-derived type,* or in a mid-series termination of a mid-shunt equivalent *m-derived type*—the derived types being obtained from a *constant-K* structure as a prototype. It is easy to show (see Sect. 17.126) that such terminations are desirable provided other filters are not to be connected either in series or in parallel with the filter under consideration—in other words, provided the filter works into terminal impedances, at each end, that are essentially pure resistances.

Consider, then, a filter terminated in a mid-shunt termination of a mid-series equivalent *m-derived type* of structure. If the prototype is a *constant-K type*, the mid-series image impedance of the prototype (see Chapter XI) is

$$Z_I = \sqrt{Z_1Z_2}\sqrt{1+\frac{1}{4}\frac{Z_1}{Z_2}} = Z_0\sqrt{1+\frac{1}{4}\frac{Z_1}{Z_2}} \tag{73}$$

In such a structure as is here under consideration, the nominal iterative impedance, Z_0, is, in general, as has been seen in Sect. 17.114, the value of the terminal impedance (assumed in this case to be pure resistance) into which the filter operates. Suppose, therefore, that such a filter is to be terminated in a resistance Z_0 and that we wish to find the value of the reflection loss existing between it and the mid-shunt image impedance Z_I' of an equivalent mid-series *m-derived type* of structure.

It is known from Chapter XI that the ratio of the mid-series image impedance to that of the mid-shunt image impedance of any symmetrical filter is

$$\frac{Z_I}{Z_I'} = 1+\frac{1}{4}\frac{Z_1}{Z_2} \tag{74}$$

But as has been seen in Sect. 16.31, the value of Z_1/Z_2 for an *m-derived type* is:

$$\left(\frac{Z_1}{Z_2}\right)_m = \frac{4m^2\left(\frac{Z_1}{Z_2}\right)_K}{(1-m^2)\left(\frac{Z_1}{Z_2}\right)_K+4} \tag{75}$$

Whence

$$1+\frac{1}{4}\left(\frac{Z_1}{Z_2}\right)_m = \frac{\left(\frac{Z_1}{Z_2}\right)_K+4}{(1-m^2)\left(\frac{Z_1}{Z_2}\right)_K+4} \tag{76}$$

in which $(Z_1/Z_2)_K$ is the impedance ratio of the *constant-K* prototype and $(Z_1/Z_2)_m$ is that of the *m-derived type*. Hence, from (74) and (76)

$$\frac{Z_I}{Z_I'} = \frac{\left(\frac{Z_1}{Z_2}\right)_K+4}{(1-m^2)\left(\frac{Z_1}{Z_2}\right)_K+4} \tag{77}$$

But since the Z_I of the derived type is the same as that of the prototype, it follows from equation (73) that

$$Z_I = Z_0\sqrt{1 + \frac{1}{4}\left(\frac{Z_1}{Z_2}\right)_K} \tag{78}$$

From (77) and (78) it is evident that

$$\frac{Z_0}{Z_I'} = \frac{\left(\frac{Z_1}{Z_2}\right)_K + 4}{\left[(1 - m^2)\left(\frac{Z_1}{Z_2}\right)_K + 4\right]\sqrt{1 + \frac{1}{4}\left(\frac{Z_1}{Z_2}\right)_K}} \tag{79}$$

If then, from the reflection factor, which is (see Sect. 11.7)

$$\frac{\sqrt{4Z_I'Z_0}}{Z_I' + Z_0} = \frac{\sqrt{4\frac{Z_0}{Z_I'}}}{1 + \frac{Z_0}{Z_I'}} \tag{80}$$

we determine the corresponding loss in napiers (which will be called A_m), the following relation is obtained:

$$\epsilon^{-A_m} = \left|\frac{\sqrt{4\frac{Z_0}{Z_I'}}}{1 + \frac{Z_0}{Z_I'}}\right| \tag{81}$$

We can, by means of (79) and (81), determine the value of the reflection loss at each terminal which must be added to the attenuation losses in order to get the total transmission loss in the filter—provided the interaction loss can be neglected.

It is noted from (79) that if

$$\left(\frac{Z_1}{Z_2}\right)_K \equiv U_K + jV_K \tag{82}$$

there are three variables, viz., U_K, V_K, and m, which must be reduced to two variables if curves of constant loss loci are to be shown on a plane. This may be done by giving one of them either a definite value or a fixed relation to one of the others. It is known that V_K has a value differing from zero only when a wave filter is dissipative. For ordinary amounts of dissipation, V_K is so small that it has but little effect upon A_m. Hence, for most of the frequency range we can put, without appreciable error, $V_K = 0$, leaving A_m as a function simply of m and U_K. At two points, however, the value of V_K exerts a large influence upon the value of A_m—these two points being the critical frequency where $U_K = -4$ and the frequency of maximum attenuation in the *m-type* where $U_K = -4/(1-m^2)$. These points are not ordinarily of sufficient interest in filter designs to warrant plotting curves for them. If desired, however, such curves can be plotted, without appreciable error, as functions of m and V_K.

Curves applying for the free transmitting range and the attenuating range (with the exception of frequencies near points of infinite attenuation) may, then, be computed from (79) and (81)—by substituting U_K for its approximate equivalent, $(Z_1/Z_2)_K$.

As an example of the computation of such curves, suppose it is desired to calculate the reflection loss when $m = .6$ and $U_K \doteqdot (Z_1/Z_2)_K = -6.00$. Then from (79)

$$\frac{Z_0}{Z_I'} = \frac{-2}{[.64(-6)+4]\sqrt{1-1.5}} = \pm j17.68 \tag{83}$$

Putting this value in (81)

$$\epsilon^{-A_m} = \left|\frac{\sqrt{4}\,(\pm j17.68)}{1 \pm j17.68}\right| = \frac{8.41}{17.71} = .474 \tag{84}$$

Whence, from exponential tables

$$A_m = .746 \text{ napier} \tag{85}$$

Such curves, then, make it possible to determine the reflection loss, (in napiers) which exists in any structure terminated in a mid-shunt termination of a mid-series equivalent *m-derived type* of structure—provided the prototype of such derived structure is a constant-K structure and that the terminal impedance into which the structure is working is equal to the nominal iterative impedance Z_0 of the filter. These last two conditions are practically always fulfilled in every design of filter and consequently do not greatly restrict the value of the curves. Since the ratio of the mid-shunt image impedance, of a mid-series equivalent *m-derived type* of filter, to Z_0 is exactly the same (assuming the same value of m) as is the ratio of Z_0 to the mid-series image impedance of the mid-shunt equivalent *m-derived type* of filter, it is evident that the reflection losses in the two cases are the same. This is, of course, on the assumption that both structures are derived from the same constant-K prototype. In other words, such terminal loss curves apply equally well either to the mid-shunt termination of the mid-series equivalent type or to the mid-series termination of the mid-shunt equivalent type of *m-derived* structures.

17.52 Filters Terminating in an X-Series Element.—In the case where two filters have cut-off frequencies relatively close together and where the two filters are to be connected together for parallel operation, it can be shown from a study of the impedance relations that it is usually desirable to terminate both filters in an x-series (or its equivalent) section of a filter of the *constant-K type*—x being the fractional part of the total series impedance, Z_1. If, for example, a filter were terminated in mid-series, x would be .5, while if the termination were in a full series arm, x would be 1.0. As a matter of fact, if two or more filters are to be connected in parallel, as referred to above, it can be shown that it is usually desirable to terminate each filter in approximately an $x = .8 \pm$ section (or its equivalent) of the *constant-K type*. The exact value of x, however, may vary under different conditions from .7 to. 9 or possibly even over a greater range.

Consequently, a problem which often comes up is to determine the loss of a filter so terminated—when connected between two pure resistance terminal impedances each of the value Z_0. In other words, it is desired to have curves showing the value of the terminal losses which, when added to the attenuation losses, will, as before, (neglecting the interaction factor) give the total transmission loss in the filter.

If the filter were to terminate in a one-half section of a *constant-K type*, it would be possible to get the corresponding reflection or terminal loss ($\equiv A_x'$) from the formulae already derived—such losses being obtainable from formula (79) by equating m to unity, thereby getting the ratio of Z_0/Z_I' for the prototype or *constant-K type*. Then

$$\frac{Z_0}{Z_I'} = \sqrt{1 + \frac{1}{4}\left(\frac{Z_1}{Z_2}\right)_K} \tag{86}$$

This relation combined with an equation of the sort given by (80), viz.,

$$\epsilon^{-A_x'} = \left| \frac{\sqrt{4\dfrac{Z_0}{Z_I'}}}{1 + \dfrac{Z_0}{Z_I'}} \right| \tag{87}$$

makes it possible to calculate the desired reflection loss ($\equiv A_x'$).

Suppose that it is desired to determine the loss of such a structure as that shown in Fig. 11. We can first determine the loss due to the attenuation proper in the filter; secondly, the reflection loss at the end of the filter which is not shown in the figure; and thirdly, by the method discussed below, the loss at the end of the filter which is shown.

If the filter had been terminated at its mid-series image impedance Z_I, the terminal loss could have been determined by means of formulae (79) and (81). In addition to the above loss, however, there is now a still further loss ($\equiv A_x''$) due to the $(x - .5)\ Z_1$ element which additional loss, by Thévenin's Theorem, is given by the relation

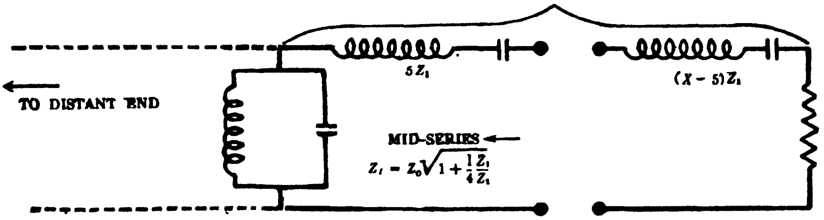

FIG. 11. x-series termination of a ladder type of structure.

$$\epsilon^{-A_x''} = \left| \frac{Z_0 + Z_I}{Z_0 + Z_I + (x - .5)Z_1} \right|$$

$$= \left| \frac{Z_0\left[1 + \sqrt{1 + \frac{1}{4}\frac{Z_1}{Z_2}}\right]}{Z_0\left[1 + \sqrt{1 + \frac{1}{4}\frac{Z_1}{Z_2}}\right] + (x - .5)Z_1} \right| \tag{88}$$

But since $Z_0^2 = Z_1 Z_2$, this becomes

$$\epsilon^{-A_x''} = \left| \frac{1 + \sqrt{1 + \frac{1}{4}\frac{Z_1}{Z_2}}}{1 + \sqrt{1 + \frac{1}{4}\frac{Z_1}{Z_2}} + (x - .5)\sqrt{\frac{Z_1}{Z_2}}} \right| \tag{89}$$

Consequently, the loss A_x'' given by (89), added to the reflection loss ($= A_x'$) corresponding to equation (87), gives the total terminal loss ($\equiv A_x$) which must be added to the total attenuation losses and to the terminal loss at the other end of the filter in order to give the total transmission loss of the filter—neglecting as before the effect of the interaction factor.

It will be observed that equation (89), as well as equation (79), involves three variables, x, U_K, and V_K (since $Z_1/Z_2 \equiv U_K + jV_K$), which, as in the case of the corresponding equation for A_m, must be reduced to two variables if curves of constant loss loci are to be shown on a plane. This may be done as before without any serious loss of approximation, except near the critical frequency, where $U_K = -4$, by neglecting the value of V_K and assuming in the above formula that $Z_1/Z_2 = U_K$. With this assumption, curves of the terminal loss A_x can be plotted in terms of U_K and x.

Let us consider an example of the above method of calculating such curves. Assume $U_K = Z_1/Z_2 = -10$ and that $x = .8$. Then from (86)—if the filter were terminated at mid-series—the impedance ratio would be

$$\frac{Z_0}{Z_I'} = \sqrt{1 - \frac{10}{4}} = \pm j1.225 \tag{90}$$

which by equation (87) gives

$$\epsilon^{-A_x'} = \left|\frac{2.21}{1 \pm j1.225}\right| = \frac{2.21}{1.58} = 1.40 \tag{91}$$

Whence, from exponential tables, $A_x' = -.336$ napier. In other words, if the filter had terminated in a mid-series termination ($x = .5$) there would have been a reflection *gain* of .336 napier to add to the other losses in the filter.

The additional loss which is due to the extra .3 end section is given by equation (89) and is

$$\begin{aligned}\epsilon^{-A_x''} &= \left|\frac{1 + \sqrt{-1.5}}{1 + \sqrt{-1.5} + .3\sqrt{-10}}\right| = \left|\frac{1 \pm j1.225}{1 \pm j1.225 \pm j.949}\right| \\ &= \left|\frac{1.58}{1 \pm j2.174}\right| = \frac{1.58}{2.39} = .661\end{aligned} \tag{92}$$

which from exponential tables indicates a loss of .414 napier for A_x''. Consequently, the total terminal loss A_x, which is the sum of A_x' and A_x'', is $.414 - .336 = .078$ napier.

17.6 Practical Methods of Filter Design.

17.61 Design Data Ordinarily Given.—The design data ordinarily given for a low pass wave filter are: (1) the maximum allowable transmission loss below a given frequency f_1; (2) the minimum allowable loss above another and higher frequency f_2; (3) the terminal conditions—impedances, etc.—between which the filter is to operate; and (4) the value of the *coil dissipation constant* Q ($\equiv L\omega/R$), *which is the ratio of the reactance to the effective resistance of the coils.* The resistance introduced by the condensers is usually considered to be negligible.

17.62 General Procedure.—If the above losses are expressed in TU or in miles of standard cable the first procedure is to reduce them to napiers by multiplying them by the proper factor. It is next seen from the formulae given in Sect. 11.8 that if the loss in the attenuated range is so large (1 or 2 napiers) that the interaction factor can be neglected, the total loss in the filter—assumed to be terminated at both ends in pure resistance impedances of the value Z_0—is composed simply of the reflection losses and the attenuation loss. It is furthermore noticed that the maximum value which the reflection factor can have, when the image impedance of the filter is a pure imaginary—which it approximates closely in the attenuated range—is a value of 2 which corresponds to a total reflection *gain*, at both ends of the circuit, of .694 napier or 6.36 miles. In the attenuated range of a filter, moreover, the reflection factor at each end of the circuit will practically always reach the maximum value of $-$ .347 napier as the negative value of U_K increases. As a result, in the preliminary design of any filter, whether it be a L.P., H.P., B.P. or L. and H.P. filter, it is always reasonably safe to assume that the minimum *transmission loss* of a filter in the attenuated range will, over a wide frequency range, be essentially .7 napier *less than* the minimum total *attenuation constant* of the filter. The problem of design then resolves itself first into a determination of what combination of sections of the various kinds that are available will give a total attenuation loss, in the attenuated range of the filter, of .7 napier in excess of that specified for the minimum transmission loss.

Before proceeding with such a design, however, it is necessary to make some assumption with regard to the theoretical cut-off frequency f_c of the filter. It is evident that due to dissipation, etc., there will always be an appreciable loss at the theoretical cut-off frequency f_c of the filter, this loss varying roughly from 5 to 15 miles in many commercial designs of filters. If, then, the loss of a low pass filter at the frequency f_1 is to be kept down to a value of 2 miles or less, it is evident that the theoretical cut-off frequency f_c must be appreciably higher than f_1. In such cases a rough method to use for determining the tentative location of the theoretical cut-off frequency of the filter is to assume it to lie approximately half way between the two given limiting frequencies, or to assume that

$$f_c = \frac{f_1 + f_2}{2} \tag{93}$$

If, then, as is usually found advisable, one of the sections of the filter is to have its point of infinite attenuation approximately at the frequency f_2, the value of a is:

$$a = \frac{f_2}{f_c} \tag{94}$$

Having the above information, the next procedure is to determine what combination of whole and half filter sections will give the necessary

L P FILTER, TERMINATED FOR PARALLEL CONNECTION WITH A H.P. FILTER

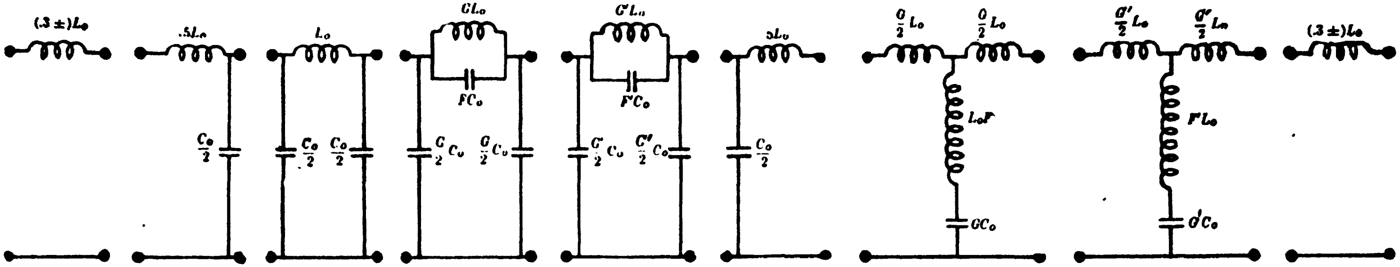
(.3 ±)L_o
.5L_o
$\frac{C_o}{2}$
L_o
$\frac{C_o}{2}$ $\frac{C_o}{2}$
GL_o
FC_o
$\frac{G}{2}C_o$ $\frac{G}{2}C_o$
$G'L_o$
$F'C_o$
$\frac{G'}{2}C_o$ $\frac{G'}{2}C_o$
.5L_o
$\frac{C_o}{2}$
$\frac{G}{2}L_o$ $\frac{G}{2}L_o$
L_oF
GC_o
$\frac{G'}{2}L_o$ $\frac{G'}{2}L_o$
$F'L_o$
$G'C_o$
(.3 ±)L_o

Fig 12A

STRUCTURE SHOWN IN Fig 12A, COMBINED INTO COMPOSITE FILTER

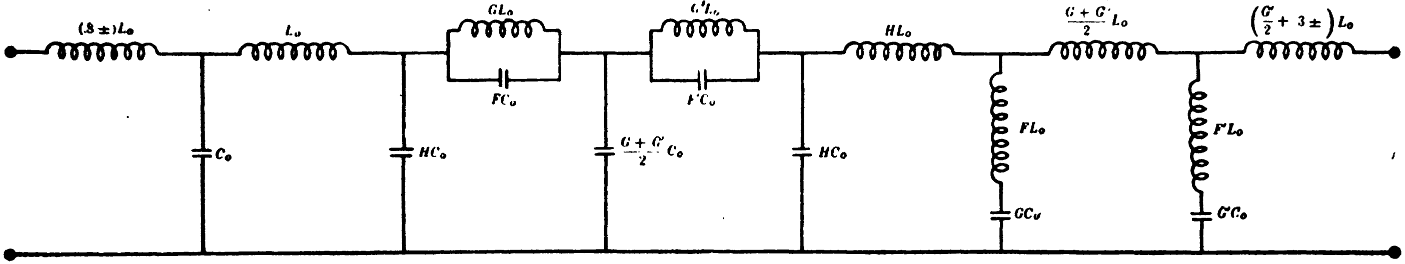

Fig. 12B

FIGS. 12A and 12B. Illustration of the method of combining two- and three-element low pass wave filter sections.

attenuation in the attenuated range. These attenuation losses may ordinarily be computed on the assumption that the structures are non-dissipative, since it is easy to show that the effects of dissipation in the coils and condensers, over a wide range of frequencies, can ordinarily be neglected. As a result, the losses so determined—after the total reflection gain of .7 napier previously referred to has been subtracted—give a fairly close approximation of the transmission loss of the filter. The only frequencies in the attenuated range where dissipation usually cannot be neglected are at the frequencies of maximum attenuation, in which points one is not ordinarily particularly interested, inasmuch as the loss at these frequencies is usually much above that existing in other portions of the attenuated range.

17.621 Combination of 2-Element and *M*-Derived Types.—From the formulae already derived in Sect. 17.4, it is evident that *the two-element type of low pass filter sections can be combined with the mid-series equivalent or mid-shunt equivalent m-derived types of sections in such a way that the resulting structure will have a total image transfer constant which is the sum of the individual image transfer constants of the component filter sections, and image impedances which will be determined solely by the characteristics of the terminal sections of the composite filter.* The detailed method of combining such two- and three-element filter sections into a composite filter structure is shown in Figs. 12A and 12B. The resulting design formulae for the various values of the elements of the composite structure are very useful and make it possible to determine readily the elements of any composite low pass filter structure comprised of two- and three-element sections. The principle of building up any number or combination of such sections should be self-evident when once the process involved in Figs. 12A and 12B is clearly understood. The process of building up any composite high pass filter comprised of two- and three-element sections is similar to that employed for any low pass filter. Examples of this are illustrated in Figs. 12C and 12D.

H.P. FILTER, TERMINATED FOR PARALLEL CONNECTION WITH A L.P. FILTER

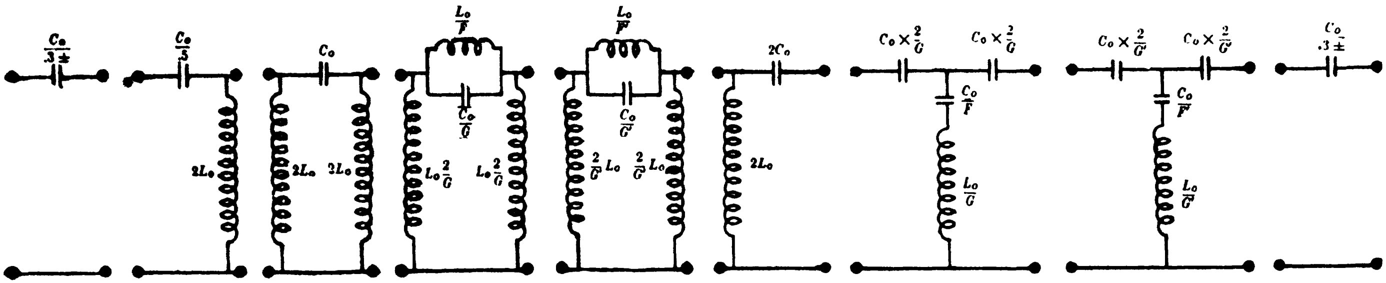

Fig. 12C

STRUCTURE SHOWN IN FIG. 12C COMBINED INTO COMPOSITE FILTER

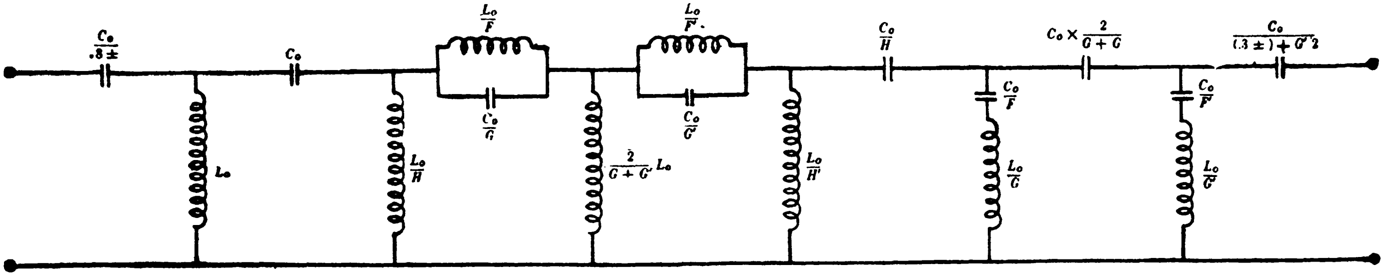

Fig. 12D

FIGS. 12C and 12D. Illustration of the method of combining two- and three-element high pass wave filter sections.

CHAPTER XVIII

CORRECTIVE NETWORKS OR EQUALIZERS*

18.0 A *corrective network* is one that is designed to have a *given loss characteristic.* Since the transmission loss caused by any structure is, in general, a function both of its iterative impedances Z_K and its propagation constant P, it is desirable, if possible, to eliminate one of the above parameters (Z_K or P) from consideration so that the loss will be a function simply of the other variable. This will be true if either of the iterative impedances of the corrective network can be made equal to that of one of the terminating impedances, since under these conditions the interaction and reflection factors will become unity, leaving the attenuation loss as the total transmission loss in the structure. When, as is frequently the case, corrective networks are used to offset or correct for the attenuation-frequency characteristics of non-loaded cables, open-wire lines, etc., they are frequently called *attenuation equalizers* or simply *equalizers.*

18.1 Constant Resistance Structures.—In most cases where a corrective network or equalizer is required, the terminal impedance, in one direction at least, is essentially a fixed pure resistance R. If, therefore, a structure is used whose iterative impedance is a pure resistance and is independent of the frequency, the transmission loss caused by such a structure may be made identical to its attenuation loss.

In Fig. 1 there are shown five general types of structures whose iterative impedance—in one direction at least—is a pure resistance and is independent of the frequency. In all of these structures it is noted that the product of Z_{11} and Z_{21} must be equal to R^2 and be independent of the frequency. Let us consider, therefore, the necessary requirements in order that such a condition may be fulfilled.

If the above relation is to be fulfilled and if Z_{11} consists of certain elements in series, then Z_{21} must consist of corresponding elements in shunt—the product of the individual impedances of the series and shunt elements always being maintained constant; i.e., independent of the frequency—and equal to R^2. This may possibly best be seen by reference to the following example based on the generalized structure shown in drawing A of Fig. 1 and exemplified, in detail, in the structure shown

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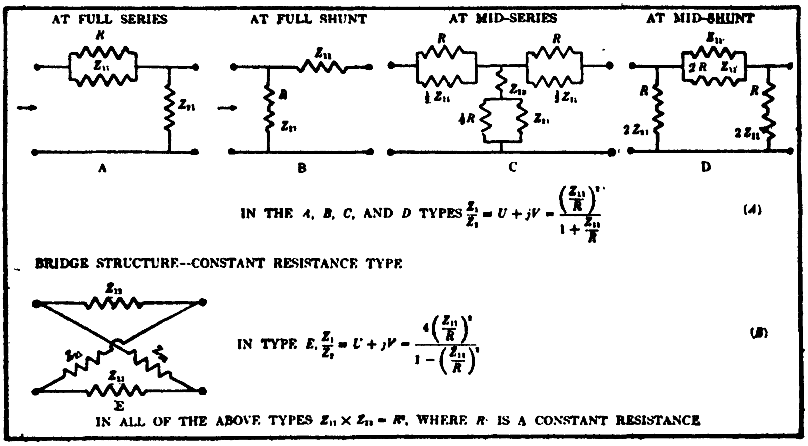
* The general types of structures discussed in this chapter are due to O. J. Zobel.


AT FULL SERIES
AT FULL SHUNT
AT MID-SERIES
AT MID-SHUNT
R
Z_{11}
Z_{21}
$\frac{1}{2}Z_{11}$
$4R$
$2R$
$2Z_{21}$
A
B
C
D
IN THE A, B, C, AND D TYPES $\frac{Z_1}{Z_2} = U + jV = \frac{\left(\frac{Z_{11}}{R}\right)^2}{1+\frac{Z_{11}}{R}}$ (A)
BRIDGE STRUCTURE--CONSTANT RESISTANCE TYPE
IN TYPE E, $\frac{Z_1}{Z_2} = U + jV = \frac{4\left(\frac{Z_{11}}{R}\right)^2}{1-\left(\frac{Z_{11}}{R}\right)^2}$ (B)
E
IN ALL OF THE ABOVE TYPES $Z_{11} \times Z_{21} = R^2$, WHERE R IS A CONSTANT RESISTANCE

Figs. 1A, 1B, 1C, 1D and 1E. General recurrent networks having constant resistance iterative impedances.

in Fig. 2—in which the elements in the series arm are related to the corresponding elements in the shunt arm as follows:

$$\frac{L_{11}}{C_{21}} = \frac{L_{12}}{C_{22}} = \frac{L_{22}}{C_{12}} = \frac{L_{13}}{C_{23}} = \frac{L_{23}}{C_{13}} = R_{11}R_{21} = R^2 \tag{1}$$

With such relations, then, the product of Z_{11} and Z_{21} will be equal to R^2 and will be independent of the frequency. The structures Z_{11} and Z_{21} are called *inverse networks of constant resistance product R^2.*

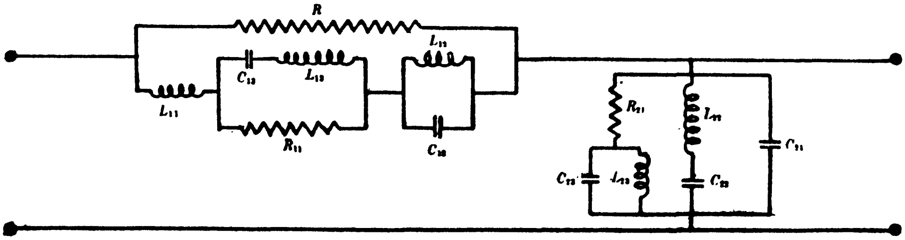


Fig. 2. Circuit illustrating the method of combining impedance elements in such a way that the product of the impedance of one arm and that of the impedance of the other arm is a constant, independent of frequency.

Assuming that one end of any structure shown in Fig. 1 is to be terminated in a pure resistance load, the problem of obtaining its transmission loss is, as has been pointed out, simply that of obtaining its attenuation loss. It is easy to show that the propagation constant of any of the structures shown in Figs. 1A, 1B, 1C or 1D is given by the relation

$$P = 2 \sinh^{-1} \tfrac{1}{2} \sqrt{\frac{\left(\dfrac{Z_{11}}{R}\right)^2}{1 + \dfrac{Z_{11}}{R}}} \tag{2}$$

Consequently, from this formula it is apparent that for any given value of Z_{11}/R, the transmission loss of such a structure can be determined. In order to facilitate the design of such structures, it is possible to derive from equation (2) the relation

$$\left(1 + \frac{R_{11}}{R}\right)^2 + \left(\frac{X_{11}}{R}\right)^2 = \epsilon^{2A} \tag{3}$$

and to plot curves, such as are given in Fig. 3, which give at once the

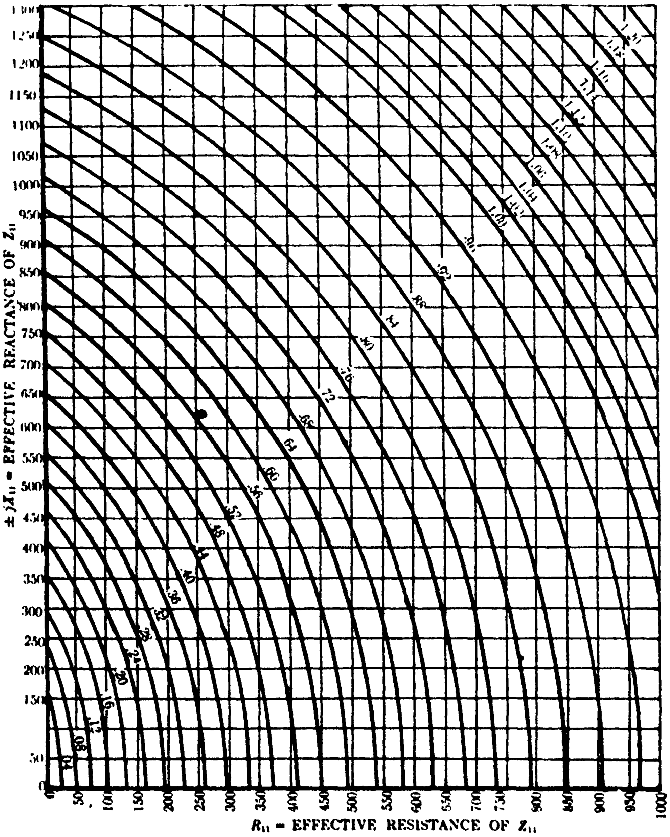


FIG. 3. Curves showing the relation between the real portion of the propagation constant, A, per section of a constant resistance structure and its series impedance $Z_{11} = R_{11} + jX_{11}$. In these curves R is assumed to be 600 ohms.

attenuation loss (in napiers) of a constant resistance structure in terms of the effective resistance R_{11} and the effective reactance X_{11} of the Z_{11} arm. Equation (3) shows that the center of these concentric circles is located at the point $-R, 0$ and that the radius of the curves is ϵ^{A}. While the curves of Fig. 3 are drawn for a specific value of the terminal resistance R, viz., 600 ohms, it is, however, evident from equation (3) that the curves are equally applicable for a design requiring any other value of R such as R_0—the procedure in this latter case being simply to multiply all the resistances and inductances in Z_{11} and Z_{21} by the ratio R_0/R and all the capacities by the ratio R/R_0.

It may be noted that the loss caused by any such constant resistance structure is identically the same as if a simple series impedance of the value $2Z_{11}$, or a simple shunt impedance of the value $\frac{1}{2}Z_{21}$, is inserted between two equal terminal impedances, each of which has the value R.

18.11 Design of Typical Corrective Network.—With the foregoing as a basis we will take a typical example of a corrective network and show in more or less detail the procedure which may be followed in its design.

Assume that it is required to design a corrective network of the constant resistance type such that when it is connected between 600 ohm terminal impedances it will have a loss-frequency characteristic that is a straight line—the loss at 21,000 cycles being 7.2 miles greater than the loss at 33,300 cycles, this latter loss being less than 1 mile. It is further to be assumed that the actual loss curve will not deviate from the above ideal curve by more than ± $\frac{1}{4}$ mile at any frequency.

In considering the general make-up of the impedance elements to be used for the Z_{11} arm of the above structure it is evident from such curves as are given in Fig. 3 that if the loss at 33.3 kilocycles is to be very small the impedance of the Z_{11} arm must also have a very small value at this frequency—the value increasing in magnitude as the frequency decreases. A consideration of the impedance-frequency characteristics of the various possible combinations of elements indicates that a combination consisting of a resonant unit paralleled by a condenser and a resistance should satisfy the general shape of the curve which it is required to meet. Assume then a general structure such as is shown in Fig. 1A and that for the Z_{11} arm a resistance element R_1 is shunted by a reactance element —this latter consisting of two condensers C_1 and C_2 and an inductance L_1 as shown in Fig. 4. In deciding on the value of R_1 let us consider what would be the loss existing at zero frequency if the straight line curve, which is to be approximated, were to be extended to that frequency. Such an extension of the straight line curve indicates that it would pass through approximately 20.4 miles or 2.23 napiers. From an inspection of curves of the type given in Fig. 3 it can be shown that the value of the resistance corresponding to this loss would be approximately 5,000 ohms. Consequently, since the impedance of the Z_{11} arm will be simply R_1 ohms at zero frequency, it is seen that, if the structure is to have the above loss at zero frequency, the value of R_1 should be 5,000 ohms.

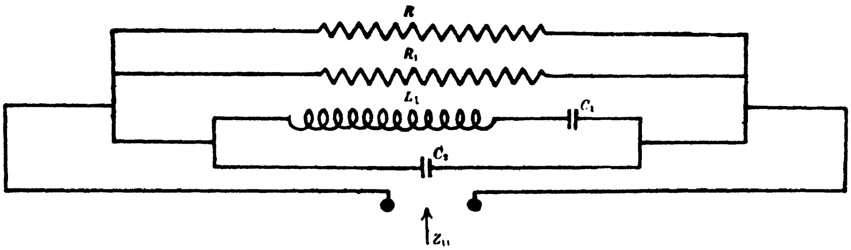


FIG. 4. Series arm for the proposed constant resistance equalizer.

Having thus more or less arbitrarily decided as to the value of the resistance R_1 to be used in the Z_{11} arm, the next matter to be determined is the make-up of the reactance element shunting this resistance. If a resistance r is to be shunted by a reactance x, the effective resistance R' and the effective reactance X' of the combination may be obtained from formulae (19) and (20) of Appendix F. From these formulae the following data are obtained for various (arbitrarily chosen) values of x shunting the assumed value of r (or R_1)—namely, 5,000 ohms.

r	x	R'	X'	A	L
5,000	− 1,000	192	− 961	.73	6.7
5,000	− 700	96	− 686	.49	4.5
5,000	− 350	24.4	− 348	.17	1.6

The values of A are the losses, in napiers, which are read from the curves of Fig. 3, while the values of L are the corresponding losses in miles of standard cable. They are the losses that would be caused by a constant resistance structure—when connected between terminal impedances of $600 \underline{/0°}$ ohms—the impedance of whose Z_{11} arm was $(R' + jX')$, or, in other words, whose Z_{11} arm consisted of a 5,000 ohm resistance r (or R_1) in parallel with a reactance x which varies as shown above.

The above table indicates that when the combined reactance x of the three elements (L_1, C_1 and C_2) is − 1,000 ohms, the loss caused by the constant resistance structure is 6.7 miles. From the arbitrary ideal curve (arbitrary simply in the fact that the minimum loss at 33,300 cycles is assumed to be .75 mile) which has been plotted in Fig. 5, a loss of 6.7 miles should occur at a frequency of 23,100 cycles. Hence, neglecting any effects of dissipation in the reactance elements, the reactance x should be $-$ 1,000 ohms at a frequency of 23,100 cycles.

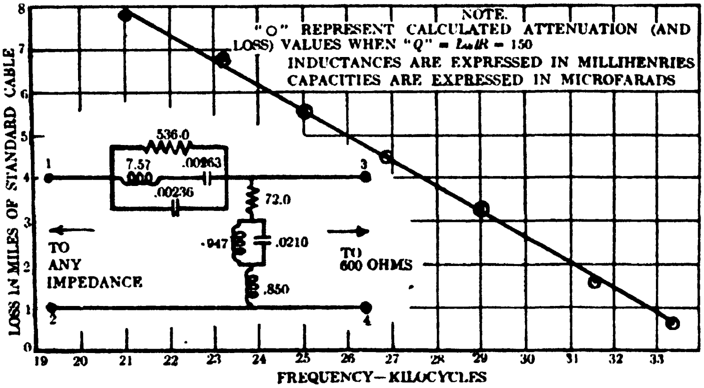


Fig. 5. Corrective network or equalizer and its loss-frequency characteristic.

Similarly, when $x = -700$ ohms it is evident from the table that the loss caused by the structure is 4.5 miles which, from the curve in Fig. 5, should occur at a frequency of 26,900 cycles. Therefore, the reactance x of the three reactive elements of the Z_{11} arm should be $-$ 700 ohms at 26,900 cycles. In a like manner, we find that a reactance x of $-$ 350 ohms, which corresponds to a loss of 1.6 miles, should occur at a frequency of approximately 31,800 cycles.

The effects of dissipation in the reactance elements will, however, be such as to increase slightly the loss caused by them at those frequencies near their resonant frequency. In other words, it is reasonable to assume that the actual loss caused by the elements, when their reactance is $-$ 350 ohms, will be slightly higher than 1.6 miles. We will, therefore, assume that, due to resistance in the coils, the value of $x = -350$ ohms should occur at a frequency of 31,600 cycles corresponding to a loss, from the curve in Fig. 5, of 1.7 miles.

From the preceding, it is evidently required to design a reactive network (see Fig. 6) composed of three elements such that its reactance x

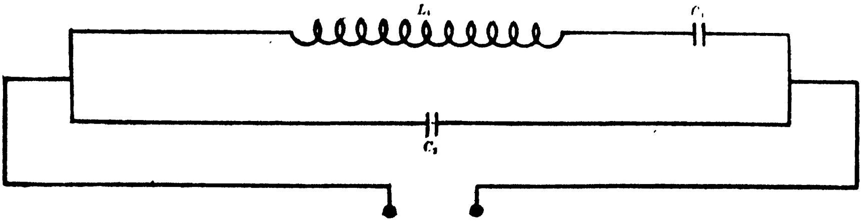


Fig. 6. Series arm of constant resistance structure consisting of a resonant combination paralleled with a condenser.

shall have the following values at the frequencies given:

Frequency	Reactance x
23,100	− 1,000
26,900	− 700
31,600	− 350

By means of formulae (28) to (32) of Appendix F, we find that the following values for the inductances and the condensers will result in a network that will have the desired reactances at the frequencies referred to in the above table, and should, therefore, give an attenuation characteristic in conformity with the original requirements:

$$L_1 = 7.57 \times 10^{-3} \text{ henry}$$
$$C_1 = .00263 \times 10^{-6} \text{ mf.}$$
$$C_2 = .00236 \times 10^{-6} \text{ mf.}$$

The Z_{11} arm will, therefore, be made up as shown in Fig. 7.

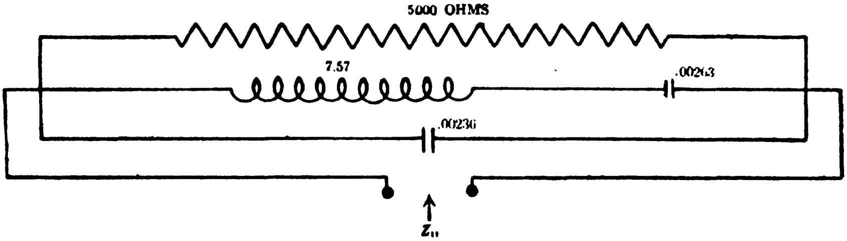


FIG. 7. Complete series arm for the constant resistance equalizer. Inductances are expressed in millihenries and capacities in microfarads.

Assuming that the ratio Q of the reactance of the coil to its effective resistance is 150 and that there is only a negligible amount of dissipation in the condensers, values are then computed for the impedances of the Z_{11} arm at different frequencies. These impedances are given in the following table:

Frequency	Z_{11}	A	L
21,000	$266 - j\,1120$	.86	7.9
23,100	$192 - j\,962$	.73	6.7
25,000	$138 - j\,820$	.61	5.6
26,900	$96 - j\,686$	.49	4.5
29,000	$64 - j\,541$	.36	3.3
31,600	$31 - j\,346$	.18	1.6
33,300	$9.4 - j\,215$	.08	.7

The values of the corresponding losses in napiers, A, were read from the curves of Fig. 3 and the corresponding values in miles L of standard cable are plotted as the points in circles in Fig. 5. It is seen that these points all lie within about .1 mile of the ideal loss curve represented by the straight line. Hence, a corrective network of the constant resistance type of structure having a Z_{11} arm as shown above will readily meet the requirements originally imposed.

If the constant resistance structure is made up in accordance with the form shown in Fig. 1A, it is then possible to combine the 5,000 ohm resistance of the Z_{11} arm with the value of R (600 ohms) since the two resistances are directly in parallel. This combination produces the resistance 536 ohms as shown in the completed structure in Fig. 5. The values of the various elements in the shunt arm Z_{21} are then obtained as previously explained in Sect. 18.1 (equation 1).

The completed structure should then produce the loss-frequency characteristic indicated by the points regardless of the impedance to which the series arm (terminals 1 and 2) is connected provided only that the shunt arm (terminals 3 and 4) is connected to the iterative impedance of the structure—which, in the case under consideration, is 600 $\underline{/0^\circ}$ ohms.

18.12 Method of Associating Additional Elements with Any Given Impedance so that the Total Combination has an Impedance Characteristic that is Approximately a Pure Resistance and Independent of the Frequency.—In addition to utilizing the constant resistance type of structure as a corrective network, that is, as a means for producing any desired loss-frequency characteristic, it is also apparent that it can be associated with any device in such a way that the complete structure will have approximately a constant resistance impedance at all frequencies.

Suppose, for example, that the damped impedance of a receiver can be represented by a resistance R_{21} in series with an inductance L_{21}, in series with a parallel combination of an inductance L_{22} and a resistance R_{22}. If an impedance-frequency curve of the receiver is given, the effective resistance and reactance of the above combination of inductances and resistances can evidently be made identical to the corresponding values for the receiver at any two desired frequencies. Formulae for doing this are given in Appendix F (No. 51 to No. 54). With this structure as a basis for the value of Z_{21} in Fig. 1B, we can determine the corresponding values of R_{11}, C_{11}, C_{12} and R_{12} by relations similar to those given in equation (1). Then the completed structure, which could be made up as shown in Fig. 8, will evidently have an impedance R that is very nearly a pure resistance and whose value is essentially constant with respect to the frequency.

For example, a structure of this sort, designed to correct the damped impedance curve of an operator's receiver, was found by Zobel to have a variation in impedance deviating from a 600 ohm pure resistance by less than 5 per cent from all frequencies from zero up to 3,000 cycles. The loss caused by the associated elements was roughly 8 miles—remaining fairly constant over the frequency range from 0 to 3,000 cycles. Notwithstanding this relatively large loss, however, such a constant resistance unit is frequently of interest in those cases where the impedance of a receiver, line or similar device must be balanced by a pure resistance or other simple network in order that the maximum possible gain may be obtained from a repeater or similar device.

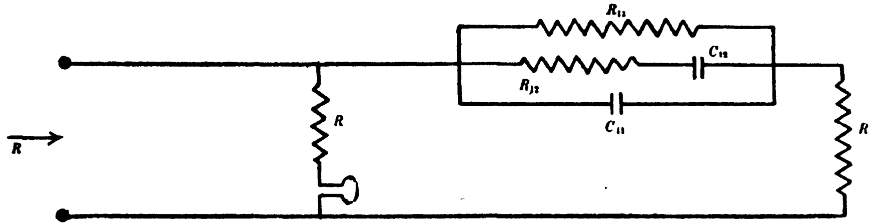


FIG. 8. Receiver and associated network—the combination giving approximately a constant pure resistance impedance, R.

18.2 Constant-K Structures.—*A constant-K structure is,* as has been already pointed out, *one of the ladder type, in which the square root of the product of its series impedance Z_1 and its shunt impedance Z_2 is equal to K—where K is a constant, independent of frequency* (or in which the product of Z_1 and Z_2 is equal to K^2). The method of making the product of two impedances Z_1 and Z_2 independent of the frequency has been explained in Sect. 18.1.

It was shown in Sect. 17.112 that the iterative impedance of a constant-K structure of the ladder type is, at mid-series

$$Z_K = Z_O\sqrt{1+\frac{1}{4}\frac{Z_1}{Z_2}} \tag{4}$$

and in Sect. 17.113 that the mid-shunt iterative impedance is

$$Z_K{}' = \frac{Z_O}{\sqrt{1+\frac{1}{4}\frac{Z_1}{Z_2}}} \tag{5}$$

where Z_0 is equal to K and is a constant that is independent of the frequency. From these equations it is evident that the impedances, Z_K or $Z_K{}'$ of any such structure will remain fairly constant and will be essentially independent of the frequency provided the value of Z_1/Z_2 does not become greater than unity or thereabouts; that is, if over a given range, Z_1/Z_2 varies from zero to unity, the iterative impedance will evidently not vary by more than 10 or 12 per cent from its nominal value, Z_0. The design of a constant-K structure is, therefore, usually based on the assumption that the make-up of each of the component sections is such that their value of Z_1/Z_2 will not become large enough to make the iterative impedance of the structure deviate sufficiently from Z_0 (or the terminal impedance) so that the interaction and reflection factors have to be taken into account.

In other words, the constant-K structure is somewhat analogous to the constant resistance structure, but is not quite as satisfactory as the latter structure when one end is working into a terminal impedance that is a constant pure resistance. This is due to the fact that the iterative impedance of the constant-K structure does not remain rigorously constant over the entire range of frequencies and, consequently, certain terminal effects cannot always be neglected. In some cases this is a serious objection while in others it is relatively unimportant.

The main advantage of a constant-K structure as compared with a constant resistance one, where but small attenuation is required, is the somewhat greater ease with which designs employing such a type of structure can be made. For example, the phase angle of Z_1 has less effect upon the attenuation characteristic of a constant-K structure than does the phase angle of Z_{11} in the case of the constant resistance structure. Consequently, in designing a constant-K structure it is, in general, only necessary to give careful consideration to the *magnitude of the effective resistance of* Z_1, whereas in the constant resistance structure the values of *both the effective resistance and the reactance* of Z_{11} must be carefully considered. With this simplification in mind, the design of a constant-K structure can be undertaken along the same general lines as has been indicated for the constant resistance structure.

CHAPTER XIX

SUPERIMPOSED CIRCUITS

19.0 The desirability as well as the possibility of simultaneously sending more than one telephone or one telegraph message over the same pair of wires—without interference—was realized many years ago. By the use of *phantom circuits* it is possible to transmit three telephone messages over two pairs of wires. By the use of a similar scheme it is possible to *simplex* a telephone circuit and thereby to obtain a non-interfering telegraph channel (employing the ground for the return) for each pair of wires that is being used for telephone purposes. Several non-interfering telegraph channels can be obtained by *compositing* a telephone circuit, while a much larger number of non-interfering telegraph messages or several telephone messages can be simultaneously superimposed on a single pair of wires by means of high frequency *carrier currents*. These various schemes are discussed briefly in the following sections.

19.1 Phantom Circuits.—Ordinarily only one telephone circuit is obtainable over each pair of wires. If, however, four wires run between two points, three telephone circuits may be obtained over these wires. Two of these telephone circuits are called *side circuits* and the other one a *phantom circuit.*

The operation of such a circuit (see Fig. 1) is as follows. Current flows from one wire C of the phantom circuit to the midpoint P_1 of the balanced winding of the phantom repeating coil No. 1. If this point

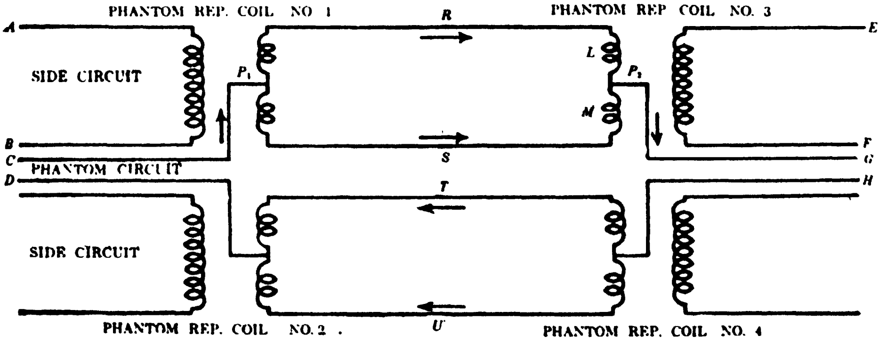


FIG. 1. Schematic diagram of a phantom circuit.

P_1 is at the electrical center of the line winding of the repeating coil and if the line wires R and S are exactly alike—i.e., if they have the same impedances—the current will split equally between the two wires R and S. At the distant end of the circuit, if P_2 is at the electrical center of phantom repeating coil No. 3, the currents will pass in equal amounts through the windings L and M and out on the phantom wire G. The return of the phantom current from wire H to wire D is by an identical process.

The phantom current will evidently not produce any interference with the side circuit, provided the former passes equally through the balanced windings L and M, etc., as under this condition no flux is produced in the core of the repeating coil and hence no electromotive force is induced in the side circuit due to current flowing in the phantom circuit. Any unbalance in impedance between the line wires R and S, or T and U, or any unbalance in the balanced windings of the phantom repeating coils, will unbalance the phantom circuit and tend to produce *crosstalk* between the phantom and the side circuits.

The *phantoming of a phantom circuit* or a *double phantom*—which is evidently possible theoretically—is usually found undesirable on account of the difficulties in properly balancing all of the eight wires.

A cable which is designed for phantom working is called a *quadded cable*. The two wires of the side circuits are first twisted together into pairs—one pair being given one length of twist and the other pair a different length of twist. These pairs, in groups of two, are then twisted into *quads*, with a still different length of twist. Such processes are necessary to secure symmetry—especially with loaded cables where the tendency to crosstalk is greater than with non-loaded cables.

It is sometimes desirable to terminate one or both of the side circuits at an intermediate point. The circuit of Fig. 2 shows one side circuit

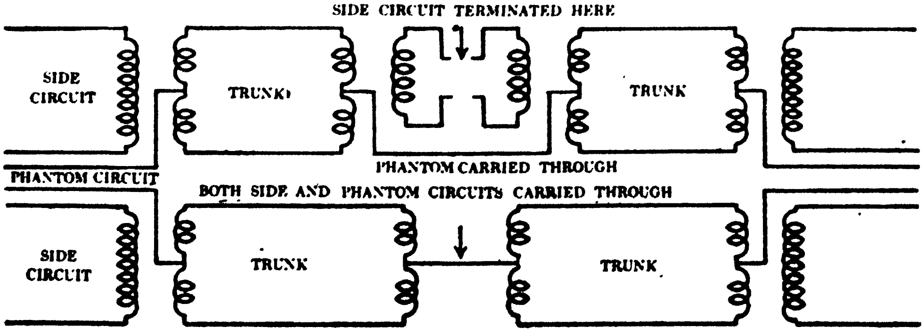


FIG. 2. Simplified diagram of a phantom circuit, with one side circuit terminating at a mid-point.

taken out at an intermediate point and the other side circuit carried through.

The method of loading a phantom circuit is shown schematically in Fig. 3. As was explained in Sect. 13.21, the side circuit loading coils add inductance only to the side circuit and the phantom coils *load* only the phantom circuit. The d-c. resistance of the phantom loading coils, however, enters into the side circuit, and hence the efficiency of the side circuit is not quite as high as that of a similar circuit which is not phantomed.

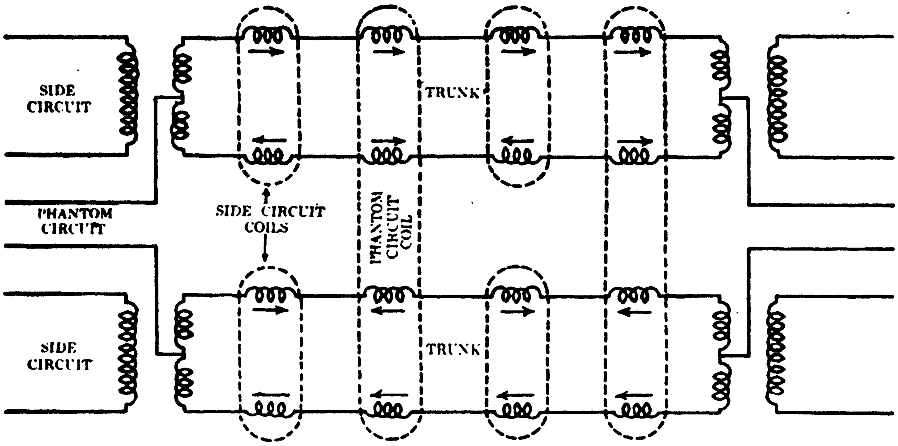


FIG. 3. Simplified diagram of a circuit loaded with side circuit and phantom circuit loading coils.

19.2 Simplex Sets.—A simplex set (see Fig. 4) consists essentially of a phantom repeating coil—the line portion of the telegraph circuit or channel being the same as that used for one side of a phantom telephone circuit.

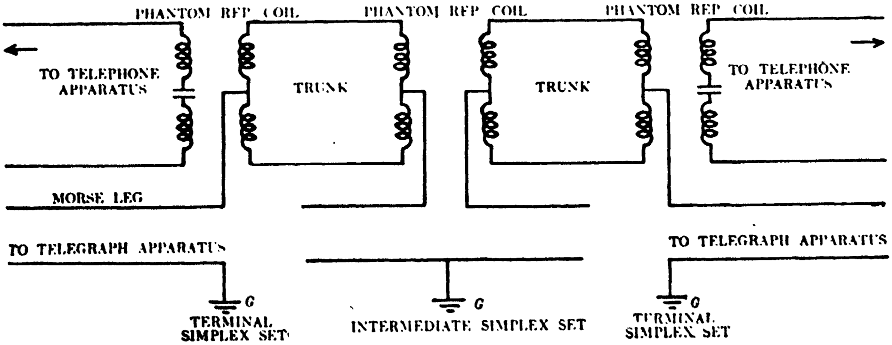


FIG. 4. Simplified diagram of a circuit using simplex sets.

The return portion of the telegraph circuit is ordinarily through the ground. If the line wires and line windings of the phantom repeating coils are properly balanced, it is evident that the telephone and telegraph currents will pass over the line simultaneously without mutual interference. The use of intermediate simplex sets makes it possible to introduce intermediate telegraph stations without interfering with through telephone service, excepting in so far as transmission and signaling losses are caused by the additional coils required.

The use of simplex sets makes it impossible to use simultaneously a phantom telephone circuit, and only gives one telegraph channel for each pair of wires.

19.3 Composite Sets.—A composite set is an apparatus arrangement whereby the telegraph currents, which involve frequencies from zero to approximately 100 cycles per second, are separated from the voice currents which contain higher frequencies. It may be regarded as a form of a wave filter.

By means of composite sets (see Fig. 5), a pair of wires normally used as a telephone line may be divided at each terminal or intermediate station so that each wire of the pair may be used as an independent telegraph circuit at the same time that the pair is used as a telephone circuit. The branches of the composite sets which are connected to the telephone apparatus are known as *telephone branches*, and those connected to the telegraph apparatus are called telegraph branches or *Morse legs*.

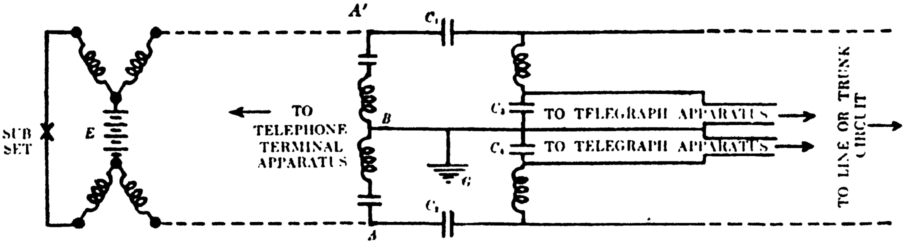


FIG. 5. Simplified diagram of a typical composite set.

Composite sets are so designed as to reduce to a minimum any possible interference between the telephone and telegraph systems due to their simultaneous operation in their respective branches. The impedance of the telegraph apparatus and that of the coil winding in series with the telegraph branch, together with the capacity of the intervening grounded condenser connected across each telegraph branch, serve to reduce the suddenness in changes of potential, due to the operation of the telegraph apparatus, at the point where each telegraph branch joins the line, thereby practically eliminating the noise known as *Morse thump* which the operation of the telegraph apparatus has a tendency to produce in the telephone circuit. The possible tendency to unbalance the circuit, due to differences in impedance between the telegraph circuits which may be connected to the two telegraph branches, is overcome by the condensers (C_3 and C_4) connected across these branches. These condensers serve to maintain the impedance of the telegraph branches to telephone currents at practically a constant value, regardless of differences in the condition of the telegraph circuits which may be connected to the two branches.

In order to reduce the momentary impulses known as *cross-fire* which pass from one telegraph branch to another through the telephone branches and to reduce the effect of these impulses upon the signaling apparatus, each telephone branch of the composite set is connected directly to ground through a path A–B (or A'–B) which has a low impedance to such impulses. The condensers in these grounded branches and in the telephone branches (C_1 and C_2) prevent interference with the proper operation of direct-current supervisory signals in cord circuits or trunks which may be connected to the line.

In order to eliminate crosstalk between phantom circuits and their side circuits, when composited, it is essential that the composite sets should not introduce any capacity or inductance unbalance into the side circuits with which they are associated. To prevent this, the condensers C_1 and C_2 are especially selected with a view to avoiding wide variations in their capacity, and the windings of the associated coils are closely balanced electrically.

Either a side circuit or a phantom circuit may be composited. To avoid unbalancing the phantom telephone circuit, however, *both* side circuits must be composited, even if there is a demand for but one or two of the four telegraph circuits which are thus rendered available.*

19.4 High-Frequency Carrier Current Telephony and Telegraphy.†—Recent developments in multiplex telephony and telegraphy have greatly increased the message-carrying capacity of long-distance telephone lines. Several telephone conversations or telegraph messages over one pair of wires are simultaneously transmitted in addition to the telephone conversation and telegraph messages provided by the ordinary methods. There is no interference between these various messages, and the subscriber is not aware that the line is being used by other subscribers at the same time. Due to the complexity and expense of the apparatus required for a system of this type, it is not practical to equip lines which are less than 150 or 200 miles in length.

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* For a more complete description of composite sets and their associated apparatus, see a paper on "Composite Telegraphy and Telephony," by J. H. Bell, *Post Office Electrical Engineer's Journal*, Vol. 15, Part 1, April, 1922, pp. 1–12.

† For a more extended discussion of the principles involved in carrier current telephony and telegraphy, see paper by E. H. Colpitts and O. B. Blackwell, *Transactions of the A. I. E. E.*, Vol. XL (1921), pp. 205–296.

In high-frequency multiplex transmission, the voice current or telegraph signals are *modulated* with a high-frequency current which *carries* them to the other end of the line. These high-frequency currents, commonly called *carrier currents*, are sustained oscillations generated by vacuum tubes. The carrier current and the voice current, or the telegraph signals, are impressed on a device known as a *modulator*. This may be a vacuum tube or any other electrical device which has a non-linear relation between input voltage and output current. In the process of *modulation** the band of frequencies representing the voice combines with the carrier current in such a way that the entire band is shifted upward in the frequency scale to a position adjacent to the carrier frequency. In a multiplex system the voice current or telegraph signal for each channel modulates a carrier current of different frequency. Each carrier frequency, then, represents an individual circuit or channel. These bands of high-frequency currents—called *side bands*†—allocated to different parts of the frequency range, are then impressed on a telephone line and transmitted to the distant station. Here they enter a system of selective networks of the wave filter type. Each wave filter is so designed that it admits only that band of high-frequency currents representing a given channel. After the high-frequency bands have thus been separated, each one is again impressed on a *detector* or *demodulator* whereby the original band of voice currents or telegraph signals is restored and transmitted to the subscriber in the usual way.

The frequency range used for carrier transmission extends approximately from 3,000 to 30,000 cycles. The lower limit is determined by the fact that the ordinary telephone conversation transmitted over the line employs frequencies up to about 2,500 cycles. The range between 2,500 and 3,000 cycles is used for effecting complete separation between the ordinary voice channel and the carrier channels. The upper frequency limit is determined largely by the increased attenuation of the

* Modulation is discussed in a paper by E. B. Craft and E. H. Colpitts on "Radio Telephony," *Proc. A. I. E. E.*, Vol. 38, Part 1, 1919, pp. 305–343.

† See article on "Relations of Carrier and Side Bands in Radio Transmission," by R. V. L. Hartley, *Proc. I. R. E.*, Vol. II, No. 1, Feb., 1923, pp. 34–55.

line and by the transposition requirements necessary to prevent crosstalk or interference at high frequencies. In this frequency range as many as four two-way telephone channels or ten duplex telegraph channels have been obtained on one pair of wires.

Carrier systems are best adapted for operation over open wire nonloaded lines. Means have not yet been developed so that long cables or ordinary loaded circuits can be made suitable for carrier transmission, the attenuation and crosstalk being too great for currents of high frequency. Phantom circuits are also considered unsuitable for carrier systems due to the extreme difficulty of maintaining a sufficient degree of balance between phantom and side circuits at high frequencies.

APPENDIX A

USEFUL MATHEMATICAL FORMULAE*

I. ALGEBRA

The binomial theorem is:

$$(A \pm B)^R = A^R \pm RA^{R-1}B + \frac{R(R-1)}{2!}A^{R-2}B^2 \pm \frac{R(R-1)(R-2)}{3!}A^{R-3}B^3 + \dots B^R \tag{1}$$

In the arithmetic series:

$$A_1 + (A_1 + D) + (A_1 + 2D) + (A_1 + 3D) + \dots$$

if A_N is any term, the sum of N terms is

$$S_N = \frac{N}{2}(A_1 + A_N) = \frac{N}{2}(2A_1 + [N-1]D) \tag{2}$$

and the value of the Nth term is

$$A_N = A_1 + (N-1)D \tag{3}$$

In the geometric series:

$$A_1 + A_1R + A_1R^2 + A_1R^3 + A_1R^4 + \dots$$

if A_N is any term, the sum of N terms is

$$S_N = \frac{A_1(R^N - 1)}{R - 1} = \frac{A_NR - A_1}{R - 1} \tag{4}$$

and the value of the Nth term is

$$A_N = A_1R^{N-1} \tag{5}$$

Solution of a Quadratic Equation.

In the quadratic equation:

$$Ax^2 + Bx + C = 0$$

we have

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \tag{6}$$

The arithmetic mean between A and B is $\frac{A+B}{2}$

The geometric mean between A and B is $\sqrt{AB}$

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* For formulae relating to complex quantities, see Appendix B.

The harmonic mean between A and B is $\dfrac{2AB}{A+B}$

It is to be noted that $\dfrac{A+B}{2} > \sqrt{AB} > \dfrac{2AB}{A+B}$

Solution of a Cubic Equation.

In the cubic equation:

$$Ay^3 + 3By^2 + 3Cy + D = 0 \tag{7}$$

Let

$$q \equiv AC - B^2 \tag{8}$$

and

$$r \equiv \frac{3ABC - A^2D}{2} - B^3 \tag{9}$$

Then the required three roots are

$$y_1 = (x_1 - B)/A \tag{10}$$

$$y_2 = (x_2 - B)/A \tag{11}$$

and

$$y_3 = (x_3 - B)/A \tag{12}$$

in which the values of x_1, x_2 and x_3 are to be obtained from one of the following equations.

Case (1). If q is $-$ and $r^2 \leqq (-q)^3$:

$$\begin{aligned} x_1 &= \pm 2\sqrt{-q}\cos\left[\frac{1}{3}\cos^{-1}\frac{\pm r}{\sqrt{(-q)^3}}\right] \\ x_2 &= \pm 2\sqrt{-q}\cos\left[\frac{1}{3}\cos^{-1}\frac{\pm r}{\sqrt{(-q)^3}} + \frac{2\pi}{3}\right] \\ x_3 &= \pm 2\sqrt{-q}\cos\left[\frac{1}{3}\cos^{-1}\frac{\pm r}{\sqrt{(-q)^3}} + \frac{4\pi}{3}\right] \end{aligned} \tag{13}$$

Case (2). If q is $-$ and $r^2 \geqq (-q)^3$:

$$\begin{aligned} x_1 &= \pm 2\sqrt{-q}\cosh\left[\frac{1}{3}\cosh^{-1}\frac{\pm r}{\sqrt{(-q)^3}}\right] \\ x_2 &= \mp \sqrt{-q}\cosh\left[\frac{1}{3}\cosh^{-1}\frac{\pm r}{\sqrt{(-q)^3}}\right] \\ &\quad + j\sqrt{-3q}\sinh\left[\frac{1}{3}\cosh^{-1}\frac{\pm r}{\sqrt{(-q)^3}}\right] \\ x_3 &= \mp \sqrt{-q}\cosh\left[\frac{1}{3}\cosh^{-1}\frac{\pm r}{\sqrt{(-q)^3}}\right] \\ &\quad - j\sqrt{-3q}\sinh\left[\frac{1}{3}\cosh^{-1}\frac{\pm r}{\sqrt{(-q)^3}}\right] \end{aligned} \tag{14}$$

Case (3). If q is $+$:

$$x_1 = \pm 2\sqrt{q}\sinh\left[\frac{1}{3}\sinh^{-1}\frac{\pm r}{\sqrt{q^3}}\right]$$

$$x_2 = \mp \sqrt{q}\sinh\left[\frac{1}{3}\sinh^{-1}\frac{\pm r}{\sqrt{q^3}}\right] + j\sqrt{3q}\cosh\left[\frac{1}{3}\sinh^{-1}\frac{\pm r}{\sqrt{q^3}}\right] \tag{15}$$

$$x_3 = \mp \sqrt{q}\sinh\left[\frac{1}{3}\sinh^{-1}\frac{\pm r}{\sqrt{q^3}}\right] - j\sqrt{3q}\cosh\left[\frac{1}{3}\sinh^{-1}\frac{\pm r}{\sqrt{q^3}}\right]$$

NOTE: In all of the formulae for x_1, x_2, and x_3, the upper of the alternative signs are to be used when r is $+$ and the lower of the alternative signs are to be used when r is $-$.

Solution of a Bi-quadratic (4th Power) Equation.

$$\text{Given}\quad x^4 + ax^3 + bx^2 + cx + d = 0$$

First find any (real) root of the cubic:

$$8\lambda^3 - 4b\lambda^2 + 2\lambda(ac - 4d) - c^2 - d(a^2 - 4b) = 0 \tag{16}$$

(For the solution of a cubic see the preceding case).

The four roots of the bi-quadratic are then given by the roots of the following two quadratic equations:

$$x^2 + x\left[\frac{a}{2} - \sqrt{\frac{a^2}{4} + 2\lambda - b}\right] + \lambda + \sqrt{\lambda^2 - d} = 0 \tag{17}$$

$$x^2 + x\left[\frac{a}{2} + \sqrt{\frac{a^2}{4} + 2\lambda - b}\right] + \lambda - \sqrt{\lambda^2 - d} = 0 \tag{18}$$

Solution of Two Simultaneous Equations.

$$\text{Given}\begin{cases} Ax + By = E \\ A_1x + B_1y = E_1 \end{cases}$$

Then:

$$x = \frac{EB_1 - E_1B}{AB_1 - A_1B} \tag{19}$$

$$y = \frac{E_1A - EA_1}{AB_1 - A_1B} \tag{20}$$

Solution of Three Simultaneous Equations.

$$\text{Given}\quad \begin{cases} Ax + By + Cz + D = 0 \\ A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$$

Then:

$$x = \frac{D(B_2C_1 - B_1C_2) + D_1(BC_2 - B_2C) + D_2(B_1C - BC_1)}{A(B_1C_2 - B_2C_1) + A_1(B_2C - BC_2) + A_2(BC_1 - B_1C)} \tag{21}$$

$$y = \frac{D(A_1C_2 - A_2C_1) + D_1(A_2C - AC_2) + D_2(AC_1 - A_1C)}{A(B_1C_2 - B_2C_1) + A_1(B_2C - BC_2) + A_2(BC_1 - B_1C)} \tag{22}$$

$$z = \frac{D(A_2B_1 - A_1B_2) + D_1(AB_2 - A_2B) + D_2(A_1B - AB_1)}{A(B_1C_2 - B_2C_1) + A_1(B_2C - BC_2) + A_2(BC_1 - B_1C)} \tag{23}$$

Solution of Four Simultaneous Equations.

$$\text{Given}\quad \begin{cases} Aw + Bx + Cy + Dz = E \\ A_1w + B_1x + C_1y + D_1z = E_1 \\ A_2w + B_2x + C_2y + D_2z = E_2 \\ A_3w + B_3x + C_3y + D_3z = E_3 \end{cases}$$

Then:

$$\begin{aligned} w = \{&E[B_1(C_2D_3 - C_3D_2) - B_2(C_1D_3 - C_3D_1) + B_3(C_1D_2 - C_2D_1)] \\ &- E_1[B(C_2D_3 - C_3D_2) - B_2(CD_3 - C_3D) + B_3(CD_2 - C_2D)] \\ &+ E_2[B(C_1D_3 - C_3D_1) - B_1(CD_3 - C_3D) + B_3(CD_1 - C_1D)] \\ &- E_3[B(C_1D_2 - C_2D_1) - B_1(CD_2 - C_2D) + B_2(CD_1 - C_1D)]\} \div \{H\} \end{aligned} \tag{24}$$

$$\begin{aligned} x = \{&E[-A_1(C_2D_3 - C_3D_2) + A_2(C_1D_3 - C_3D_1) - A_3(C_1D_2 - C_2D_1)] \\ &+ E_1[A(C_2D_3 - C_3D_2) - A_2(CD_3 - C_3D) + A_3(CD_2 - C_2D)] \\ &+ E_2[-A(C_1D_3 - C_3D_1) + A_1(CD_3 - C_3D) - A_3(CD_1 - C_1D)] \\ &+ E_3[A(C_1D_2 - C_2D_1) - A_1(CD_2 - C_2D) + A_2(CD_1 - C_1D)]\} \div \{H\} \end{aligned} \tag{25}$$

$$\begin{aligned} y = \{&E[-A_1(B_3D_2 - B_2D_3) + A_2(B_3D_1 - B_1D_3) - A_3(B_2D_1 - B_1D_2)] \\ &+ E_1[A(B_3D_2 - B_2D_3) + A_2(BD_3 - B_3D) - A_3(BD_2 - B_2D)] \\ &+ E_2[A(B_1D_3 - B_3D_1) - A_1(BD_3 - B_3D) - A_3(B_1D - BD_1)] \\ &+ E_3[A(B_2D_1 - B_1D_2) - A_1(B_2D - BD_2) + A_2(B_1D - BD_1)]\} \div \{H\} \end{aligned} \tag{26}$$

$$\begin{aligned} z = \{&E[-A_1(B_2C_3 - B_3C_2) + A_2(B_1C_3 - B_3C_1) - A_3(B_1C_2 - B_2C_1)] \\ &+ E_1[A(B_2C_3 - B_3C_2) + A_2(B_3C - BC_3) - A_3(B_2C - BC_2)] \\ &+ E_2[A(B_3C_1 - B_1C_3) - A_1(B_3C - BC_3) - A_3(BC_1 - B_1C)] \\ &+ E_3[A(B_1C_2 - B_2C_1) - A_1(BC_2 - B_2C) + A_2(BC_1 - B_1C)]\} \div \{H\} \end{aligned} \tag{27}$$

In all of the above equations the value of H is

$$\begin{aligned} H = {} & A[B_1(C_2D_3 - C_3D_2) - B_2(C_1D_3 - C_3D_1) + B_3(C_1D_2 - C_2D_1)] \\ & - A_1[B(C_2D_3 - C_3D_2) - B_2(CD_3 - C_3D) + B_3(CD_2 - C_2D)] \\ & + A_2[B(C_1D_3 - C_3D_1) - B_1(CD_3 - C_3D) + B_3(CD_1 - C_1D)] \\ & - A_3[B(C_1D_2 - C_2D_1) - B_1(CD_2 - C_2D) + B_2(CD_1 - C_1D)] \end{aligned}$$

II. HYPERBOLIC FUNCTIONS*

$$\sinh u = \frac{e^u - e^{-u}}{2} = -\sinh(-u) = \frac{1}{\operatorname{csch} u} = \tan (GD)u \tag{28}$$

$$\sinh u = \frac{2\tanh\frac{u}{2}}{1-\tanh^2\frac{u}{2}} = \frac{\tanh u}{\sqrt{1-\tanh^2 u}} = \sqrt{\cosh^2 u - 1} \tag{29}$$

$$= u + \frac{u^3}{3!} + \frac{u^5}{5!} + \frac{u^7}{7!} + \cdots$$

$$\cosh u = \frac{e^u + e^{-u}}{2} = \cosh(-u) = \frac{1}{\operatorname{sech} u} = \sec (GD)u \tag{30}$$

$$\cosh u = \frac{1+\tanh^2\frac{u}{2}}{1-\tanh^2\frac{u}{2}} = \frac{1}{\sqrt{1-\tanh^2 u}} = \sqrt{\sinh^2 u + 1} \tag{31}$$

$$= 1 + \frac{u^2}{2!} + \frac{u^4}{4!} + \frac{u^6}{6!} + \cdots$$

$$\tanh u = \frac{e^u - e^{-u}}{e^u + e^{-u}} = -\tanh(-u) = \frac{1}{\coth u} = \frac{\sinh u}{\cosh u} \tag{32}$$

$$= \sin (GD)u$$

$$\tanh u = \sqrt{1-\operatorname{sech}^2 u} = u - \frac{u^3}{3} + \frac{2u^5}{15} - \frac{17u^7}{315} + \frac{62u^9}{2835} - \cdots \tag{33}$$

$$\text{if} \quad u^2 < \left(\frac{\pi}{2}\right)^2$$

$$\sinh(a \pm b) = \sinh a \cosh b \pm \cosh a \sinh b \tag{34}$$

$$\cosh(a \pm b) = \cosh a \cosh b \pm \sinh a \sinh b \tag{35}$$

$$\tanh(a \pm b) = \frac{\tanh a \pm \tanh b}{1 \pm \tanh a \tanh b} \tag{36}$$

$$\tanh\left(\frac{a \pm b}{2}\right) = \frac{\sinh a \pm \sinh b}{\cosh a + \cosh b} \tag{37}$$

$$\sinh a + \sinh b = 2\sinh\frac{a+b}{2}\cosh\frac{a-b}{2} \tag{38}$$

$$\sinh a - \sinh b = 2\sinh\frac{a-b}{2}\cosh\frac{a+b}{2} \tag{39}$$

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* For additional relations between anti-hyperbolic functions see Appendix C. For formulae relating to complex quantities see Appendix B.

$$\cosh a + \cosh b = 2 \cosh \frac{a+b}{2} \cosh \frac{a-b}{2} \tag{40}$$

$$\cosh a - \cosh b = 2 \sinh \frac{a+b}{2} \sinh \frac{a-b}{2} \tag{41}$$

$$\tanh a \pm \tanh b = \frac{\sinh (a \pm b)}{\cosh a \cosh b} \tag{42}$$

$$(\sinh a + \cosh a)^u = \cosh ua + \sinh ua \tag{43}$$

$$\begin{gathered} \sinh \frac{a}{2} = \sqrt{\frac{1}{2}(\cosh a - 1)}; \qquad \cosh \frac{a}{2} = \sqrt{\frac{1}{2}(\cosh a + 1)}; \\ \tanh \frac{a}{2} = \frac{\cosh a - 1}{\sinh a} = \frac{\sinh a}{\cosh a + 1} \end{gathered} \tag{44}$$

$$\sinh 2a = 2 \sinh a \cosh a = \frac{2 \tanh a}{1 - \tanh^2 a} \tag{45}$$

$$\cosh 2a = \cosh^2 a + \sinh^2 a = 2 \cosh^2 a - 1 = 2 \sinh^2 a + 1 \tag{46}$$

$$\tanh 2a = \frac{2 \tanh a}{1 + \tanh^2 a} \tag{47}$$

$$\operatorname{csch} a = \frac{1}{a} - \frac{a}{6} + \frac{7a^3}{360} - \frac{31a^5}{15,120} + \cdots \quad \text{if} \quad a^2 < \pi^2 \tag{48}$$

$$\operatorname{sech} a = 1 - \frac{a^2}{2} + \frac{5a^4}{24} - \frac{61a^6}{720} + \cdots \quad \text{if} \quad a^2 < \left(\frac{\pi}{2}\right)^2 \tag{49}$$

$$\coth a = \frac{1}{a} + \frac{a}{3} - \frac{a^3}{45} + \frac{2a^5}{945} - \frac{a^7}{4725} + \cdots \quad \text{if} \quad a^2 < \pi^2 \tag{50}$$

$$\begin{aligned} \sinh^{-1} a &= \log_e (a + \sqrt{a^2 + 1}) = \cosh^{-1} \sqrt{a^2 + 1} \\ &= a - \frac{a^3}{6} + \frac{3a^5}{40} - \frac{5a^7}{112} + \frac{35a^9}{1152} - \quad \text{if} \quad a^2 < 1 \end{aligned} \tag{51}$$

$$\begin{aligned} \cosh^{-1} a &= \log_e (a + \sqrt{a^2 - 1}) = \sinh^{-1} \sqrt{a^2 - 1} \\ &= \log_e 2a - \frac{1}{4a^2} - \frac{3}{32a^4} - \frac{5}{96a^6} - \quad \text{if} \quad a^2 > 1 \end{aligned} \tag{52}$$

$$\begin{aligned} \tanh^{-1} a &= \frac{1}{2} \log_e \frac{1 + a}{1 - a} = \cosh^{-1} \frac{1}{\sqrt{1 - a^2}} = \sinh^{-1} \frac{a}{\sqrt{1 - a^2}} \\ &= a + \frac{a^3}{3} + \frac{a^5}{5} + \frac{a^7}{7} + \cdots \quad \text{if} \quad a^2 < 1 \end{aligned} \tag{53}$$

$$\operatorname{csch}^{-1} a = \log_e \frac{1 + \sqrt{a^2 + 1}}{a}$$

$$= \sinh^{-1}\frac{1}{a} = \frac{1}{a} - \frac{1}{6a^3} + \frac{3}{40a^5} - \frac{5}{112a^7} - \quad \text{if} \quad a^2 > 1 \tag{54}$$

$$\operatorname{sech}^{-1} a = \log_e \frac{1 + \sqrt{1 - a^2}}{a} = \cosh^{-1}\frac{1}{a}$$

$$= \log_e \frac{2}{a} - \frac{a^2}{4} - \frac{3a^4}{32} - \frac{5a^6}{96} - \quad \text{if} \quad a^2 < 1 \tag{55}$$

$$\coth^{-1} a = \frac{1}{2}\log_e \frac{a+1}{a-1} = \tanh^{-1}\frac{1}{a} = \frac{1}{a} + \frac{1}{3a^3} + \frac{1}{5a^5} + \frac{1}{7a^7} + \quad \text{if} \quad a^2 > 1 \tag{56}$$

III. APPROXIMATION FORMULAE

In these approximation formulae δ is used to signify an extremely small quantity and L to signify an extremely large quantity—when compared with the other quantities with which they are associated. The sign ÷ is used to denote "equals approximately."

$$(1 \pm \delta)^N \div 1 \pm N\delta \quad \text{provided} \quad N > 1 \tag{57}$$

$$\frac{1}{(1 \pm \delta)^N} \div 1 \mp N\delta \quad \text{provided} \quad N > 1 \tag{58}$$

$$\sqrt{A(A + \delta)} \div A + \frac{\delta}{2} \tag{59}$$

$$\sqrt{A \pm \delta} \div \sqrt{A} \pm \frac{\delta}{2\sqrt{A}} \tag{60}$$

$$\sin (x \pm \delta) \div \sin x \pm \delta \cos x \tag{61}$$

$$\cos (x \pm \delta) \div \cos x \mp \delta \sin x \tag{62}$$

$$\tan (x \pm \delta) \div \tan x \pm \frac{\delta}{\cos^2 x} \tag{63}$$

$$\log_e (x \pm \delta) \div \log_e x \pm \frac{\delta}{x} - \frac{1}{2}\left(\frac{\delta}{x}\right)^2 \tag{64}$$

$$\log_e (1 \pm \delta) \div \pm \delta - \tfrac{1}{2}\delta^2 \tag{65}$$

$$\sinh (x \pm \delta) \div \sinh x \pm \delta \cosh x \tag{66}$$

$$\cosh (x \pm \delta) \div \cosh x \pm \delta \sinh x \tag{67}$$

$$\tanh (x \pm \delta) \div \frac{\tanh x}{1 \pm \delta \tanh x} \tag{68}$$

$$\sin\delta \doteq \delta \tag{69}$$

$$\cos\delta \doteq 1 \tag{70}$$

$$\tan\delta \doteq \delta \tag{71}$$

$$\sin^{-1}\delta \doteq \delta \tag{72}$$

$$\cos^{-1}\delta \doteq \frac{\pi}{2}(2K-1) \tag{73}$$

where K is any integer

$$\tan^{-1}\delta \doteq \delta \tag{74}$$

$$\sinh\delta \doteq \delta \tag{75}$$

$$\cosh\delta \doteq 1 \tag{76}$$

$$\tanh\delta \doteq \delta \tag{77}$$

$$\sinh^{-1}\delta \doteq \delta \tag{78}$$

$$\tanh^{-1}\delta \doteq \delta \tag{79}$$

$$\sin(\delta \pm jB) \doteq \delta\cosh B \pm j\sinh B \tag{80}$$

$$\cos(\delta \pm jB) \doteq \cosh B \mp j\delta\sinh B \tag{81}$$

$$\tan(\delta \pm jB) \doteq \frac{2\delta \pm j\sinh 2B}{1+\cosh 2B} \tag{82}$$

$$\sinh(\delta \pm jB) \doteq \delta\cos B \pm j\sin B \tag{83}$$

$$\cosh(\delta \pm jB) \doteq \delta\cos B \pm j\delta\sin B \tag{84}$$

$$\tanh(\delta \pm jB) \doteq \frac{2\delta \pm j\sin 2B}{1+\cos 2B} \tag{85}$$

$$\sinh L \doteq \cosh L \doteq \tfrac{1}{2}e^{L} \tag{86}$$

$$\tanh L \doteq 1 \tag{87}$$

$$\sin(A \pm jL) \doteq \cosh L\,\underline{/\pm\tan^{-1}(\cot A)} \doteq \tfrac{1}{2}e^{L}\,\underline{\Big/\pm\left(\frac{\pi}{2}-A\right)} \tag{88}$$

$$\cos(A \pm jL) \doteq \cosh L\,\underline{/\mp\tan^{-1}(\tan A)} \doteq \tfrac{1}{2}e^{L}\,\underline{/\mp A} \tag{89}$$

$$\tan(A \pm jL) \doteq \tanh 2L\,\underline{/\pm 90^\circ} \tag{90}$$

$$\sinh(L \pm jB) \doteq \sinh L\,\underline{/\pm B} \doteq \tfrac{1}{2}e^{L}\,\underline{/\pm B} \tag{91}$$

$$\cosh(L \pm jB) \doteq \tfrac{1}{2}e^{L}\,\underline{/\pm B} \doteq \tfrac{1}{2}\log_{10}^{-1}(.4343L)\,\underline{/\pm B} \tag{92}$$

$$\tanh(L \pm jB) \doteq 1 \tag{93}$$

$$\sinh(A \pm j\delta) \doteq \sinh A \pm j\delta\cosh A \tag{94}$$

$$\cosh(A \pm j\delta) \doteq \cosh A \pm j\delta\sinh A \tag{95}$$

$$\tanh(A \pm j\delta) \doteq \frac{\tanh A \pm j\delta}{1 \pm j\delta\tanh A} \tag{96}$$

$$e^{\delta} \doteq 1 + \delta \tag{97}$$

$$e^{-\delta} \doteq 1 - \delta \tag{98}$$

IV. TRIGONOMETRIC FORMULAE

$$\sin A = \frac{1}{\csc A} = \sqrt{1 - \cos^2 A} = \frac{\tan A}{\sqrt{1 + \tan^2 A}}$$

$$= -\sin(-A) = A - \frac{A^3}{3!} + \frac{A^5}{5!} - \frac{A^7}{7!} + \cdots \qquad (99)$$

$$\cos A = \frac{1}{\sec A} = \sqrt{1 - \sin^2 A} = \frac{1}{\sqrt{1 + \tan^2 A}}$$

$$= \cos(-A) = 1 - \frac{A^2}{2!} + \frac{A^4}{4!} - \frac{A^6}{6!} + \cdots \qquad (100)$$

$$\tan A = \frac{1}{\cot A} = \frac{\sin A}{\cos A} = \frac{\sin A}{\sqrt{1 - \sin^2 A}} = \frac{\sqrt{1 - \cos^2 A}}{\cos A}$$

$$= -\tan(-A) = A + \frac{A^3}{3} + \frac{2A^5}{15} + \frac{17A^7}{315} \qquad (101)$$

$$+ \cdots \text{ if } A^2 < \left(\frac{\pi}{2}\right)^2$$

$$\sin 2A = 2 \sin A \cos A = \frac{2 \tan A}{1 + \tan^2 A} \qquad (102)$$

$$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2 \sin^2 A = 2 \cos^2 A - 1 \qquad (103)$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A} \qquad (104)$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \qquad (105)$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \qquad (106)$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \qquad (107)$$

$$\sin^{-1} A = \cos^{-1} \sqrt{1 - A^2} = \tan^{-1} \frac{A}{\sqrt{1 - A^2}}$$

$$= 2 \cos^{-1} \sqrt{\frac{1 + \sqrt{1 - A^2}}{2}} = 2 \sin^{-1} \sqrt{\frac{1 - \sqrt{1 - A^2}}{2}}$$

$$= 2 \tan^{-1} \frac{1 - \sqrt{1 - A^2}}{A} = -\sin^{-1}(-A) \qquad (108)$$

$$= \tfrac{1}{2} \cos^{-1}(1 - 2A^2) = \tfrac{1}{2} \sin^{-1} 2A\sqrt{1 - A^2}$$

$$= \tfrac{1}{2} \tan^{-1} \frac{2A\sqrt{1 - A^2}}{1 - 2A^2}$$

$$= A + \frac{A^3}{6} + \frac{3A^5}{40} + \frac{5A^7}{112} + \cdots \text{ if } A^2 < 1$$

$$\begin{aligned}\cos^{-1} A &= \sin^{-1}\sqrt{1-A^2} = \tan^{-1}\frac{\sqrt{1-A^2}}{A} = 2\cos^{-1}\sqrt{\frac{1+A}{2}} \\ &= 2\sin^{-1}\sqrt{\frac{1-A}{2}} = 2\tan^{-1}\sqrt{\frac{1-A}{1+A}} = \tfrac{1}{2}\cos^{-1}(2A^2-1) \\ &= \tfrac{1}{2}\sin^{-1} 2A\sqrt{1-A^2} = \tfrac{1}{2}\tan^{-1}\frac{2A\sqrt{1-A^2}}{2A^2-1} \\ &= \pi - \cos^{-1}(-A) = \frac{\pi}{2} - A - \frac{A^3}{6} - \frac{3A^5}{40} - \frac{5A^7}{112} \\ &\qquad - \cdots \text{ if } A^2 < 1\end{aligned} \tag{109}$$

$$\begin{aligned}\tan^{-1} A &= \sin^{-1}\frac{A}{\sqrt{1+A^2}} = \cos^{-1}\frac{1}{\sqrt{1+A^2}} = \tfrac{1}{2}\tan^{-1}\frac{2A}{1-A^2} \\ &= \tfrac{1}{2}\sin^{-1}\frac{2A}{1+A^2} = \tfrac{1}{2}\cos^{-1}\frac{1-A^2}{1+A^2} = 2\sin^{-1}\sqrt{\frac{\sqrt{1+A^2}-1}{2\sqrt{1+A^2}}} \\ &= 2\cos^{-1}\sqrt{\frac{1+\sqrt{1+A^2}}{2\sqrt{1+A^2}}} = 2\tan^{-1}\frac{\sqrt{1+A^2}-1}{A} \\ &= -\tan^{-1}(-A) = A - \frac{A^3}{3} + \frac{A^5}{5} - \frac{A^7}{7} + \cdots \text{ if } A^2 < 1\end{aligned} \tag{110}$$

$$\tan^{-1} A = \frac{\pi}{2} - \frac{1}{A} + \frac{1}{3A^3} - \frac{1}{5A^5} + \cdots \text{ if } A^2 > 1 \tag{111}$$

V. DIFFERENTIAL CALCULUS

$$d(\sin x) = \cos x dx \tag{112}$$

$$d(\cos x) = -\sin x dx \tag{113}$$

$$d(\tan x) = \sec^2 x dx \tag{114}$$

$$d(\cot x) = -\csc^2 dx \tag{115}$$

$$d(\sin^{-1} x) = \frac{\pm dx}{\sqrt{1-x^2}}\,{}^{*} \tag{116}$$

$$d(\cos^{-1} x) = \frac{\mp dx}{\sqrt{1-x^2}}\,{}^{\dagger} \tag{117}$$

$$d(\tan^{-1} x) = \frac{dx}{1+x^2} \tag{118}$$

$$d(\cot^{-1} x) = \frac{-dx}{1+x^2} \tag{119}$$

$$d(\sinh x) = \cosh x dx \tag{120}$$

$$d(\cosh x) = \sinh x dx \tag{121}$$

$$d(\tanh x) = \operatorname{sech}^2 x dx \tag{122}$$

$$d(\coth x) = -\operatorname{csch}^2 x dx \tag{123}$$

$$d(\sinh^{-1} x) = \frac{dx}{\sqrt{x^2+1}} \tag{124}$$

$$d(\cosh^{-1} x) = \frac{dx}{\sqrt{x^2-1}} \tag{125}$$

$$d(\tanh^{-1} x) = \frac{dx}{1-x^2} \tag{126}$$

$$d(\coth^{-1} x) = \frac{dx}{1-x^2} \tag{127}$$

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* Use the + sign, if $\sin^{-1} x$ lies in the first or fourth quadrants and use the − sign, if $\sin^{-1} x$ lies in the second or third quadrants.

† Use the − sign, if $\cos^{-1} x$ lies in the first or fourth quadrants and use the + sign, if $\cos^{-1} x$ lies in the second or third quadrants.

APPENDIX B

COMPLEX QUANTITIES

A complex quantity is a two-dimensional number. It may be represented, (1) in the general case, simply by any convenient symbol, such as P, Z, etc., or it may be represented, (2) in polar coordinates, by an expression such as $Z\,/\Theta$—in which case the quantity (Z) preceding the angular sign $(/\underline{\quad})$ is to be regarded simply as a numeric and not as a complex quantity in itself, or (3) in rectangular coordinates, a complex quantity may be represented by an expression such as $A + jB$, where j is an operator denoting 90° rotation and is defined as $\sqrt{-1}$. When a symbol or any other expression represents simply a numeric (as contrasted to a vector quantity), it is frequently enclosed in vertical bars, viz., $|Z|$, to indicate that regard is only to be given to the numerical or absolute magnitude of the quantity so enclosed. Positive angles are assumed to lie in an anti-clockwise direction of rotation and are represented by the symbol $(/\quad)$, while negative angles are assumed to lie in a clockwise direction and are indicated by the symbol $(/\overline{\quad})$. In the following discussion Θ is the angle XOP.

One complex quantity is said to be the conjugate of another when their real parts are equal and their imaginary parts are equal in magnitude but have opposite signs, viz., $A + jB$ and $A - jB$ are conjugate quantities. Expressed in polar coordinates, $Z\,/\Theta$ and $Z\,/\Theta$ are conjugate quantities. In the general symbolic system a dash over the symbol is used to indicate that one quantity is the conjugate of another, viz., $\overline{Z}$ is the conjugate of Z.

The following relations are useful in changing from the polar to the rectangular system of coordinates and vice versa (see Fig. 1).

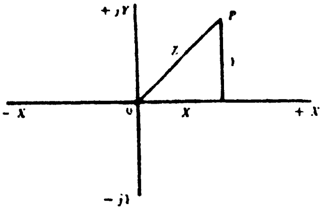

Fig. 1.—Graphical representation of a complex quantity by means of rectangular or polar coordinates.

$$X = Z\cos\Theta \qquad (1) \qquad\qquad Y = Z\sin\Theta \qquad (2)$$

$$\Theta = \tan^{-1}\frac{Y}{X} \qquad (3) \qquad\qquad \Theta = \cot^{-1}\frac{X}{Y} \qquad (4)$$

$$Z = X \div \cos\Theta \quad (5) \qquad\qquad Z = Y \div \sin\Theta \quad (6)$$

Let $P_1 \equiv X_1 + jY_1 \equiv Z_1 \underline{/\theta_1}$ and $P_2 \equiv X_2 + jY_2 \equiv Z_2 \underline{/\theta_2}$ represent two complex quantities.

Then their sum is given by the expression:

$$P_1 + P_2 = (X_1 + X_2) + j(Y_1 + Y_2) \tag{7}$$

Therefore, to add vectors, add all the real parts to obtain the real part of the resultant and add all the imaginary parts to obtain the imaginary part of the resultant.

The difference of the two quantities is given by a similar expression:

$$P_1 - P_2 = (X_1 - X_2) + j(Y_1 - Y_2) \tag{8}$$

Their product is:

$$P_1P_2 = Z_1Z_2 \underline{/\theta_1 + \theta_2} \tag{9}$$

Hence, the product of vectors is found by multiplying their scalar magnitudes or moduli and adding their angles.*

Their quotient is:

$$\frac{P_1}{P_2} = \frac{Z_1}{Z_2} \underline{/\theta_1 - \theta_2} \tag{10}$$

or is obtained by dividing one modulus by the other and by subtracting the angle or argument of the divisor from that of the dividend.*

The Nth power of the vector quantity P is:

$$P^N = Z^N \underline{/N\theta} \tag{11}$$

and its Nth root is:

$$P^{1/N} = Z^{1/N} \underline{/\theta/N} \tag{12}$$

In the above equations the complex quantities are expressed in the form in which experience has shown them to be best adapted, as a rule, for performing the processes in question. If a quantity on which it is wished to perform any of these operations is expressed in the other form than the one indicated, it will be usually found more convenient in carrying out the operation to convert the quantity, by means of equations (1) to (6), into the proper form.

If Y is large as compared with X, equation (6) *will give more accurate results than equation* (5). *If X is large as compared with Y, then equation* (5) *will give the more accurate results.*

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*The product and quotient of two complex quantities may also be obtained by the use of equations (14) and (15).

TYPICAL NUMERICAL EXAMPLES

Addition:

Given the two complex quantities $P_1 = 12 + j29$ and $P_2 = 7 - j2$, to find their sum. From equation (7)

$$P_1 + P_2 = (12 + 7) + j(29 - 2) = 19 + j27$$

Subtraction:

Given the same two quantities as in the preceding example, to find their difference. From equation (8)

$$P_1 - P_2 = (12 - 7) + j(29 + 2) = 5 + j31$$

Multiplication:

Given the same two quantities, to find their product:

From eq. (3)	From eq. (4)
$\tan \Theta_1 = \frac{Y_1}{X_1} = \frac{29}{12} = 2.417$	$\cot \Theta_2 = \frac{X_2}{Y_2} = \frac{7}{2} = 3.5$
From trigonometric tables	From trigonometric tables
$\Theta_1 = \underline{/67^\circ\, 31'}$	$\Theta_2 = \overline{/15^\circ\, 56'}$
and	and
$\sin \Theta_1 = .924$	$\cos \Theta_2 = .9616$
From eq. (6)	From eq. (5)
$Z_1 = \frac{Y_1}{\sin \Theta_1} = \frac{29}{.924} = 31.4$	$Z_2 = \frac{X_2}{\cos \Theta_2} = \frac{7}{.9616} = 7.28$
Whence	Whence
$P_1 = 31.4\, \underline{/67^\circ\, 31'}$	$P_2 = 7.28\, \overline{/15^\circ\, 56'}$

From eq. (9)

$$\begin{aligned} P_1P_2 &= Z_1Z_2\, \underline{/\Theta_1 + \Theta_2} \\ &= 31.4 \times 7.28\, \underline{/67^\circ\, 31' - 15^\circ\, 56'} \\ &= 228.6\, /51^\circ\, 35' \end{aligned}$$

From trigonometric tables

$$\cos 51^\circ\, 35' = .6214 \qquad \text{and} \qquad \sin 51^\circ\, 35' = .7835$$

From eqs. (1) and (2)

$$\begin{aligned} X &= Z \cos \Theta = 228.6 \times .6214 = 142 \\ Y &= Z \sin \Theta = 228.6 \times .7835 = 179 \end{aligned}$$

Therefore $P_1P_2 = 228.6\, \underline{/51^\circ\, 35'} = 142 + j179$

Division:

Given the same two quantities as in the preceding example, namely, $P_1 = 12 + j29$ and $P_2 = 7 - j2$, to find their quotient. In the preceding example it was found that, expressed in polar coordinates,

$$P_1 = 31.4\ /67^\circ\ 31'$$

and that

$$P_2 = 7.28\ \overline{/15^\circ\ 56'}$$

Hence, from equation (10)

$$\frac{P_1}{P_2} = \frac{31.4}{7.28}\ /67^\circ\ 31' + 15^\circ\ 56'$$

$$= 4.31\ /83^\circ\ 27'$$

It should be noted from Fig. 1 that, since the square of the hypothenuse (Z) of a right angle triangle is equal to the sum of the squares of the other two sides (X and Y), *the absolute or scalar magnitude of any complex quantity is equal to the square root of the sum of the squares of its real and imaginary components.* In other words if

$$Z\ /\theta \equiv R + jX, \qquad \text{then} \qquad |Z| = \sqrt{R^2 + X^2} \tag{13}$$

FORMULAE FOR COMPLEX QUANTITIES

For reference purposes, there is given below a list of some of the more important formulae involving complex quantities. In the following formulae the symbol K is used to denote any integer.

$$(A + jB)(C + jD) = (AC - BD) + j(BC + AD) \tag{14}$$

$$\frac{A + jB}{C + jD} = \frac{AC + BD}{C^2 + D^2} + j\frac{BC - AD}{C^2 + D^2} \tag{15}$$

$$\frac{1}{A + jB} = \frac{A}{A^2 + B^2} - j\frac{B}{A^2 + B^2} \tag{16}$$

$$\sqrt{A + jB} = \pm\left[\sqrt{\frac{\sqrt{A^2 + B^2} + A}{2}} + j\sqrt{\frac{\sqrt{A^2 + B^2} - A}{2}}\right] \tag{17}$$

$$\sqrt{A - jB} = \pm\left[\sqrt{\frac{\sqrt{A^2 + B^2} + A}{2}} - j\sqrt{\frac{\sqrt{A^2 + B^2} - A}{2}}\right] \tag{18}$$

$$\sqrt[N]{A + jB} = (A^2 + B^2)^{1/2N}\ \Big/ \underline{\frac{\tan^{-1}\left(\frac{B}{A}\right)}{N}} \tag{19}$$

$$\log_e (A \pm jB) = \tfrac{1}{2} \cdot \log_e (A^2 + B^2) + j\left[\tan^{-1}\left(\frac{B}{A}\right) \pm 2\pi K\right] \cdots \text{ if } A \text{ is } + \tag{20}$$

$$\log_e (A \pm jB) = \tfrac{1}{2} \log_e (A^2 + B^2) + j\left[\tan^{-1}\left(\frac{B}{A}\right) \pm (2K + 1)\pi\right] \cdots \text{ if } A \text{ is } - \tag{21}$$

$$\log_e A \,/\!\underline{\theta} = \log_e A + j(\theta + 2\pi K) \tag{22}$$

$$\log_e \frac{A + jB}{A - jB} = j2\left[\tan^{-1}\left(\frac{B}{A}\right) \pm 2\pi K\right] \tag{23}$$

$$\log_e (+ 1) = \pm j2\pi K \tag{24}$$

$$\log_e (- 1) = \pm j\pi(2K + 1) \tag{25}$$

$$\log_e (+ j) = \pm j\pi(2K + \tfrac{1}{2}) \tag{26}$$

$$\log_e (- j) = \pm j\pi(2K + {}^{3}) \tag{27}$$

$$A \,/\!\underline{\theta} = A \cos \theta + jA \sin \theta = Ae^{j\theta} = Ae^{j(\theta + 2\pi K)} \tag{28}$$

$$e^{j2\pi K} = 1 \tag{29}$$

$$e^{j\pi\left(2K + \frac{1}{2}\right)} = j \tag{30}$$

$$e^{j\pi(2K+1)} = - 1 \tag{31}$$

$$e^{j\pi\left(2K + \frac{3}{2}\right)} = - j \tag{32}$$

$$e^{jA} = \cos A + j \sin A \tag{33}$$

$$e^{-jA} = \cos A - j \sin A \tag{34}$$

$$e^{j2A} = \frac{1 + j \tan A}{1 - j \tan A} \tag{35}$$

$$e^{A+jB} = e^A \,/\!\underline{B} = e^A \cos B + je^A \sin B \tag{36}$$

$$\left|e^{A+jB}\right| = e^A \tag{37}$$

$$C^{A+jB} = C^A[\cos (B \log_e C) + j \sin (B \log_e C)] \tag{38}$$

$$\sin jA = j \sinh A = j\tfrac{1}{2}(e^A - e^{-A}) = - j \sinh (- A) = j\sqrt{\cosh^2 A - 1} \tag{39}$$

$$\cos jA = \cosh A = \tfrac{1}{2}(e^A + e^{-A}) = \cosh (- A) = \frac{1}{\sqrt{1 - \tanh^2 A}} \tag{40}$$

$$\tan jA = j \tanh A = j\frac{e^A - e^{-A}}{e^A + e^{-A}} = - j \tanh (- A) = j\frac{\sinh A}{\cosh A} \tag{41}$$

$$\sin A = -j\tfrac{1}{2}(e^{jA} - e^{-jA}) = -j \sinh jA \tag{42}$$

$$\cos A = \tfrac{1}{2}(e^{jA} + e^{-jA}) = \cosh jA \tag{43}$$

$$\tan A = -j\frac{e^{j2A} - 1}{e^{j2A} + 1} = -j \tanh jA \tag{44}$$

$$\begin{aligned}\sin(A \pm jB) &= \sin A \cosh B \pm j \cos A \sinh B \\ &= \pm j \sinh(B \mp jA)\end{aligned} \tag{45}$$

$$\cos(A \pm jB) = \cos A \cosh B \mp j \sin A \sinh B = \cosh(B \mp jA) \tag{46}$$

$$\tan(A \pm jB) = \frac{\sin 2A \pm j \sinh 2B}{\cos 2A + \cosh 2B} \tag{47}$$

$$(\cos A \pm j \sin B)^N = \cos NA \pm j \sin NA \tag{48}$$

$$\sqrt[N]{\cos A \pm j \sin A} = \cos\frac{A + 2\pi K}{N} \pm j \sin\frac{A + 2\pi K}{N} \tag{49}$$

$$\sin^{-1} A = -j \sinh^{-1} jA = -j \log_e (jA + \sqrt{1 - A^2}) \tag{50}$$

$$\cos^{-1} A = -j \cosh^{-1} A = -j \log_e (A + j\sqrt{1 - A^2}) \tag{51}$$

$$\tan^{-1} A = -j \tanh^{-1} jA = j\tfrac{1}{2}[\log_e (1 - jA) - \log_e (1 + jA)] \tag{52}$$

$$\sin^{-1} jA = j \sinh^{-1} A = j \log_e (A + \sqrt{1 + A^2}) \tag{53}$$

$$\cos^{-1} jA = -j \cosh^{-1} jA = \frac{\pi}{2} - j \log_e (A + \sqrt{1 + A^2}) \tag{54}$$

$$\tan^{-1} jA = j \tanh^{-1} A = j\tfrac{1}{2}[\log_e (1 + A) - \log_e (1 - A)] \tag{55}$$

$$\sinh jA = j \sin A \tag{56}$$

$$\cosh jA = \cos A \tag{57}$$

$$\tanh jA = j \tan A \tag{58}$$

$$\sinh j\pi K = 0 \tag{59}$$

$$\cosh j\pi K = \cos \pi K = (-1)^K \tag{60}$$

$$\tanh j\pi K = 0 \tag{61}$$

$$\sinh\left(\pm A + j\frac{\pi}{2}\right) = j \cosh A \tag{62}$$

$$\sinh\left(A \pm j\frac{\pi}{2}\right) = \pm j \cosh A \tag{63}$$

$$\sinh(A + j\pi K) = (-1)^K \sinh A \tag{64}$$

$$\sinh(A + j\tfrac{3}{2}\pi) = -j \cosh A \tag{65}$$

$$\sinh(A + j2\pi) = \sinh A \tag{66}$$

$$\cosh\left(\pm A + j\frac{\pi}{2}\right) = \pm j \sinh A \tag{67}$$

$$\cosh\left(A \pm j\frac{\pi}{2}\right) = \pm j \sinh A \tag{68}$$

$$\cosh(A + j\pi K) = (-1)^K \cosh A \tag{69}$$

$$\cosh(A + j\tfrac{3}{2}\pi) = -j \sinh A \tag{70}$$

$$\cosh(A + j2\pi) = \cosh A \tag{71}$$

$$\tanh\left(A \pm j\frac{\pi}{2}\right) = \coth A \tag{72}$$

$$\tanh(A + j\pi K) = \tanh A \tag{73}$$

$$\sinh j\frac{\pi}{2}(2K+1) = j \sin(2K+1)\frac{\pi}{2} = \pm j \tag{74}$$

$$\cosh j\frac{\pi}{2}(2K+1) = 0 \tag{75}$$

$$\begin{aligned} \sinh(A \pm jB) &= \sinh A \cos B \pm j \cosh A \sin B \\ &= \pm j \sin(B \mp jA) \\ &= \sqrt{\sinh^2 A + \sin^2 B}\ \underline{/\pm \tan^{-1}(\coth A \tan B)} \\ &= \sqrt{\cosh^2 A - \cos^2 B}\ \underline{/\pm \tan^{-1}(\coth A \tan B)} \end{aligned} \tag{76}$$

$$\begin{aligned} \cosh(A \pm jB) &= \cosh A \cos B \pm j \sinh A \sin B = \cos(B \mp jA) \\ &= \sqrt{\sinh^2 A + \cos^2 B}\ \underline{/\pm \tan^{-1}(\tanh A \tan B)} \\ &= \sqrt{\cosh^2 A - \sin^2 B}\ \underline{/\pm \tan^{-1}(\tanh A \tan B)} \end{aligned} \tag{77}$$

$$\tanh(A \pm jB) = \frac{\tanh A \pm j \tan B}{1 \pm j \tanh A \tan B} = \frac{\sinh 2A \pm j \sin 2B}{\cosh 2A + \cos 2B} \tag{78}$$

$$\begin{aligned} &\sinh^{-1}(A \pm jB) \\ &\quad = \sinh^{-1}\left[\sqrt{\frac{A^2 + B^2 - 1 + \sqrt{(A^2 + B^2 - 1)^2 + 4A^2}}{2}}\right] \\ &\qquad\qquad \pm j \sin^{-1}\left[\frac{\sqrt{A^2 + (B+1)^2} - \sqrt{A^2 + (B-1)^2}}{2}\right] \\ &\quad = \cosh^{-1}\left[\frac{\sqrt{A^2 + (B+1)^2} + \sqrt{A^2 + (B-1)^2}}{2}\right] \\ &\qquad\qquad \pm j \sin^{-1}\left[\frac{2B}{\sqrt{A^2 + (B+1)^2} + \sqrt{A^2 + (B-1)^2}}\right] \end{aligned} \tag{79}$$

$$\cosh^{-1}(A \pm jB)$$

$$\begin{aligned} &= \cosh^{-1}\left[\frac{\sqrt{B^2+(1+A)^2}+\sqrt{B^2+(1-A)^2}}{2}\right] \\ &\qquad \pm j\cos^{-1}\left[\frac{\sqrt{B^2+(1+A)^2}-\sqrt{B^2+(1-A)^2}}{2}\right] \\ &= \sinh^{-1}\left[\sqrt{\frac{A^2+B^2-1+\sqrt{(A^2+B^2-1)^2+4B^2}}{2}}\right] \\ &\qquad \pm j\cos^{-1}\left[\frac{2A}{\sqrt{B^2+(1+A)^2}+\sqrt{B^2+(1-A)^2}}\right] \end{aligned} \tag{80}$$

$$\tanh^{-1}(A \pm jB)$$

$$\begin{aligned} &= \tfrac{1}{2}\tanh^{-1}\frac{2A}{1+A^2+B^2}+j\tfrac{1}{2}\tan^{-1}\frac{\pm 2B}{1-A^2-B^2} \\ &= \tfrac{1}{4}\log_e\frac{B^2+(1+A)^2}{B^2+(1-A)^2}+j\tfrac{1}{2}\left[\tan^{-1}\frac{\pm B}{1+A}+\tan^{-1}\frac{\pm B}{1-A}\right] \end{aligned} \tag{81}$$

$$\sinh^{-1}(jB) = \cosh^{-1}B + j^{(2K+1)} \times \frac{2K+1}{2}\pi, \quad \text{if} \quad B > 1 \tag{82}$$

$$\sinh^{-1}(jB) = j[(-1)^K \sin^{-1}B + K\pi], \quad \text{if} \quad B < 1 \tag{83}$$

$$\cosh^{-1}(jB) = \pm\left[\sinh^{-1}B + j^{(2K+1)} \times \frac{2K+1}{2}\pi\right] \tag{84}$$

$$\tanh^{-1}(jB) = j[\tan^{-1}B + K\pi] = j\left[\frac{2K+1}{2}\pi - \cot^{-1}B\right] \tag{85}$$

$$\begin{aligned} \sin^{-1}(A \pm jB) &= \sin^{-1}\left[\frac{\sqrt{B^2+(1+A)^2}-\sqrt{B^2+(1-A)^2}}{2}\right] \\ &\qquad \pm j\cosh^{-1}\left[\frac{\sqrt{B^2+(1+A)^2}+\sqrt{B^2+(1-A)^2}}{2}\right] \end{aligned} \tag{86}$$

$$\begin{aligned} \cos^{-1}(A \pm jB) &= \cos^{-1}\left[\frac{\sqrt{B^2+(1+A)^2}-\sqrt{B^2+(1-A)^2}}{2}\right] \\ &\qquad \mp j\cosh^{-1}\left[\frac{\sqrt{B^2+(1+A)^2}+\sqrt{B^2+(1-A)^2}}{2}\right] \end{aligned} \tag{87}$$

$$\begin{aligned} \tan^{-1}(A \pm jB) &= \left[\frac{\pi-\tan^{-1}\left(\dfrac{A}{\pm B-1}\right)+\tan^{-1}\left(\dfrac{A}{\pm B+1}\right)}{2}\right] \\ &\qquad \pm j\tfrac{1}{4}\log_e\frac{A^2+(1\pm B)^2}{A^2+(1\mp B)^2} \end{aligned} \tag{88}$$

APPENDIX C

RELATIONS BETWEEN VARIOUS ANTI-HYPERBOLIC FUNCTIONS

Proofs of the Relations Existing Between Various Anti-Hyperbolic Functions

(Used in Equation (7), Chapter XI)

Given that, by definition:*

$$\cosh P \equiv \tfrac{1}{2}(e^{P} + e^{-P}) \quad (1) \qquad \sinh P \equiv \tfrac{1}{2}(e^{P} - e^{-P}) \tag{2}$$

$$\tanh P \equiv \frac{\sinh P}{\cosh P} \tag{3}$$

To prove:

$$\log_e (U + \sqrt{U^2 - 1}) = \cosh^{-1} U$$

Let $P \equiv \cosh^{-1} U$. Then

$$U = \cosh P \equiv \tfrac{1}{2}(e^{P} + e^{-P})$$

$$U^2 - 1 = \tfrac{1}{4}(e^{P} + e^{-P})^2 - 1 = \tfrac{1}{4}(e^{2P} - 2 + e^{-2P}) = \tfrac{1}{4}(e^{P} - e^{-P})^2$$

$$U + \sqrt{U^2 - 1} = \tfrac{1}{2}(e^{P} + e^{-P}) + \tfrac{1}{2}(e^{P} - e^{-P}) = e^{P}$$

Hence:

$$\log_e (U + \sqrt{U^2 - 1}) = \log_e e^{P} = P = \cosh^{-1} U \tag{4}$$

To prove:

$$P = \sinh^{-1} \sqrt{U^2 - 1} \qquad \text{where} \qquad P = \cosh^{-1} U$$

$$U = \cosh P \equiv \tfrac{1}{2}(e^{P} + e^{-P}), \qquad U^2 = \tfrac{1}{4}(e^{2P} + 2 + e^{-2P})$$

$$U^2 - 1 = \tfrac{1}{4}(e^{2P} - 2 + e^{-2P}) = \tfrac{1}{4}(e^{P} - e^{-P})^2$$

Hence:

$$\sqrt{U^2 - 1} = \tfrac{1}{2}(e^{P} - e^{-P}) \equiv \sinh P \qquad \text{or} \qquad P = \sinh^{-1} \sqrt{U^2 - 1} \tag{5}$$

To prove:

$$P = \tanh^{-1} \sqrt{\frac{U^2 - 1}{U^2}} \qquad \text{where} \qquad P = \cosh^{-1} U = \sinh^{-1} \sqrt{U^2 - 1}$$

Since $\sinh P = \sqrt{U^2 - 1}$ and $\cosh P = U$

Then:

$$\frac{\sinh P}{\cosh P} \equiv \tanh P = \sqrt{\frac{U^2 - 1}{U^2}} \qquad \text{or} \qquad P = \tanh^{-1} \sqrt{\frac{U^2 - 1}{U^2}} \tag{6}$$

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* The symbol ≡ is used to denote "equals, by definition."

To prove:

$$P = 2\sinh^{-1}\sqrt{\frac{U-1}{2}} \qquad \text{where} \qquad P = \cosh^{-1} U$$

$$\cosh P = U = \tfrac{1}{2}(e^P + e^{-P}) \qquad \text{or} \qquad \frac{U-1}{2} = \tfrac{1}{4}(e^P - 2 + e^{-P})$$

Hence:

$$\sqrt{\frac{U-1}{2}} = \tfrac{1}{2}(e^{P/2} - e^{-(P/2)}) \equiv \sinh\frac{P}{2} \quad \text{or} \quad P = 2\sinh^{-1}\sqrt{\frac{U-1}{2}} \tag{7}$$

To prove:

$$P = 2\cosh^{-1}\sqrt{\frac{U+1}{2}} \qquad \text{where} \qquad P = \cosh^{-1} U$$

$$P = \cosh^{-1} U \qquad \text{or} \qquad U = \cosh P \equiv \tfrac{1}{2}(e^P + e^{-P})$$

$$\frac{U+1}{2} = \tfrac{1}{4}(e^P + 2 + e^{-P}), \qquad \sqrt{\frac{U+1}{2}} = \tfrac{1}{2}(e^{P/2} + e^{-(P/2)}) \equiv \cosh\frac{P}{2}$$

Hence:

$$P = 2\cosh^{-1}\sqrt{\frac{U+1}{2}} \tag{8}$$

To prove:

$$P = 2\tanh^{-1}\sqrt{\frac{U-1}{U+1}}$$

where

$$P = 2\sinh^{-1}\sqrt{\frac{U-1}{2}} = 2\cosh^{-1}\sqrt{\frac{U+1}{2}}$$

Since:

$$\sinh\frac{P}{2} = \sqrt{\frac{U-1}{2}} \qquad \text{and} \qquad \cosh\frac{P}{2} = \sqrt{\frac{U+1}{2}}$$

Then:

$$\frac{\sinh\frac{P}{2}}{\cosh\frac{P}{2}} = \sqrt{\frac{U-1}{U+1}} \equiv \tanh\frac{P}{2}$$

or

$$P = 2\tanh^{-1}\sqrt{\frac{U-1}{U+1}} \tag{9}$$

APPENDIX D

FORMULAE FOR EQUIVALENT NETWORKS

I. RELATIONS BETWEEN GENERALIZED TWO-TERMINAL NETWORKS HAVING IDENTICALLY EQUIVALENT IMPEDANCE CHARACTERISTICS

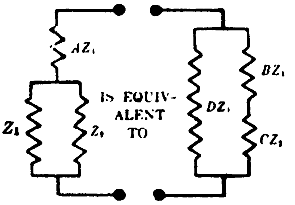


FIG. 1. Equivalent impedance networks composed of *one* combination of three generalized impedances.

The two networks shown in Fig. 1 are equivalent when:

$$B = A(1 + A) \quad (1) \qquad C = (1 + A)^2 \quad (2) \qquad D = 1 + A \quad (3)$$

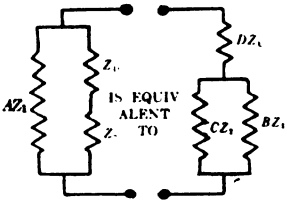


FIG. 2. Equivalent impedance networks composed of a *second* combination of three generalized impedances.

The two networks shown in Fig. 2 are equivalent when:

$$B = \frac{A^2}{1 + A} \quad (4) \qquad C = \left(\frac{A}{1 + A}\right)^2 \quad (5) \qquad D = \frac{A}{1 + A} \quad (6)$$

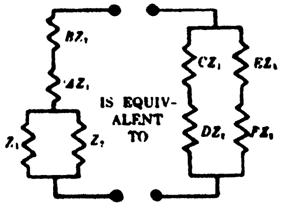


FIG. 3. Equivalent impedance networks composed of *one* combination of four generalized impedances.

The two networks shown in Fig. 3 are equivalent when:

$$C = \frac{N(M + N)}{M + N - 2B} \quad (7) \qquad D = \frac{2BN}{M + N - 2B} \quad (8)$$

$$E = \frac{N(M - N)}{N - M + 2B} \quad (9) \qquad F = \frac{2BN}{N - M + 2B} \quad (10)$$

where $M = 1 + A + B$ (11) and $N = \sqrt{M^2 - 4AB}$ (12)

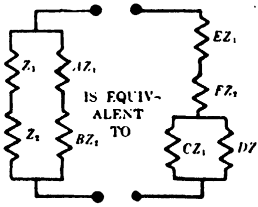


FIG. 4. Equivalent impedance networks composed of a *second* combination of four generalized impedances.

The two networks shown in Fig. 4 are equivalent when:

$$C = \frac{(A - B)^2}{(1 + A)(1 + B)^2} \quad (13) \qquad D = \frac{(A - B)^2}{(1 + A)^2(1 + B)} \quad (14)$$

$$E = \frac{A}{1 + A} \quad (15) \qquad F = \frac{B}{1 + B} \quad (16)$$

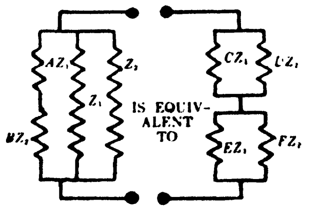


FIG. 5. Equivalent impedance networks composed of a *third* combination of four generalized impedances.

The two networks shown in Fig. 5 are equivalent when:

$$C = \frac{(2B - M + N)(M + N)}{4BN} \quad (17) \qquad D = \frac{2B - M + N}{2N} \quad (18)$$

$$E = \frac{(M + N - 2B)(M - N)}{4BN} \quad (19) \qquad F = \frac{M + N - 2B}{2N} \quad (20)$$

where:

$$M = 1 + A + B \qquad (21) \quad \text{and} \quad N = \sqrt{M^2 - 4AB} \qquad (22)$$

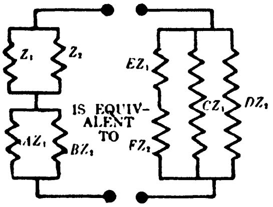


FIG. 6. Equivalent impedance networks composed of a *fourth* combination of four generalized impedances.

The two networks shown in Fig. 6 are equivalent when:

$$C = 1 + A \qquad (23) \qquad E = \frac{A(1 + A)(1 + B)^2}{(A - B)^2} \qquad (24)$$

$$D = 1 + B \qquad (25) \qquad F = \frac{B(1 + A)^2(1 + B)}{(A - B)^2} \qquad (26)$$

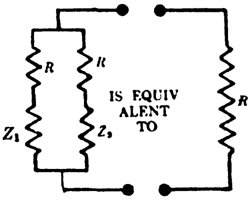


FIG. 7. Network which contains generalized impedances, Z_1 and Z_2, of a constant resistance product and whose impedance is a pure resistance, R, at all frequencies.

The two networks shown in Fig. 7 are equivalent when:

$$Z_1 Z_2 = R^2 \qquad (27)$$

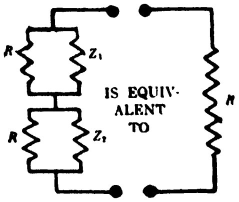


FIG. 8. A *second* type of network which contains generalized impedances, Z_1 and Z_2, of a constant resistance product and whose impedance is a pure resistance, R, at all frequencies.

The two networks shown in Fig. 8 are equivalent when:

$$Z_1Z_2 = R^2 \qquad (28)$$

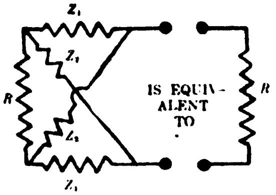


FIG. 9. Bridge type of network which contains generalized impedances, Z_1 and Z_2, of a constant resistance product and whose impedance is a pure resistance, R, at all frequencies.

The two networks shown in Fig. 9 are equivalent when:

$$Z_1Z_2 = R^2 \qquad (29)$$

II. RELATIONS BETWEEN TWO-TERMINAL STRUCTURES COMPOSED OF PURE REACTANCES AND HAVING IDENTICALLY EQUIVALENT IMPEDANCE CHARACTERISTICS

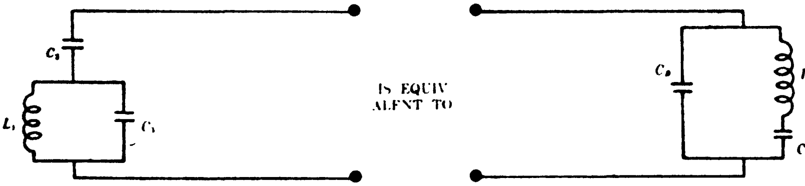


FIG. 10. Three-element equivalent networks, each containing one inductance and two capacities.

The two networks shown in Fig. 10 are equivalent when:

$$C_2 = C_A + C_B \quad (30) \qquad\qquad C_1 = \frac{C_B}{C_A}(C_A + C_B) \qquad (31)$$

$$L_1 = \frac{L_A}{\left(1 + \dfrac{C_B}{C_A}\right)^2} \qquad (32)$$

or when:

$$C_A = \frac{C_2^2}{C_1 + C_2} \quad (33) \qquad\qquad C_B = \frac{C_1C_2}{C_1 + C_2} \qquad (34)$$

$$L_A = L_1\left(1 + \frac{C_1}{C_2}\right)^2 \qquad (35)$$

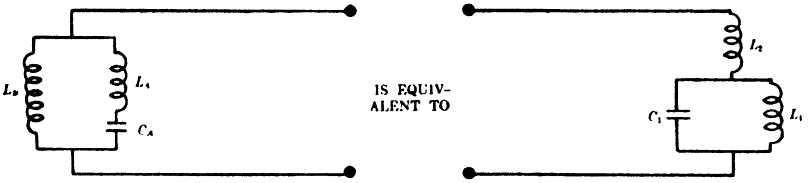


FIG. 11. Three-element equivalent networks, each containing one capacity and two inductances.

The two networks shown in Fig. 11 are equivalent when:

$$L_1 = \frac{L_B^2}{L_A + L_B} \quad (36) \qquad C_1 = C_A\left(1 + \frac{L_A}{L_B}\right)^2 \quad (37)$$

$$L_2 = \frac{L_A L_B}{L_A + L_B} \quad (38)$$

or when:

$$L_A = \frac{L_2(L_1 + L_2)}{L_1} \quad (39) \qquad C_A = \frac{C_1}{\left(1 + \frac{L_2}{L_1}\right)^2} \quad (40)$$

$$L_B = L_1 + L_2 \quad (41)$$

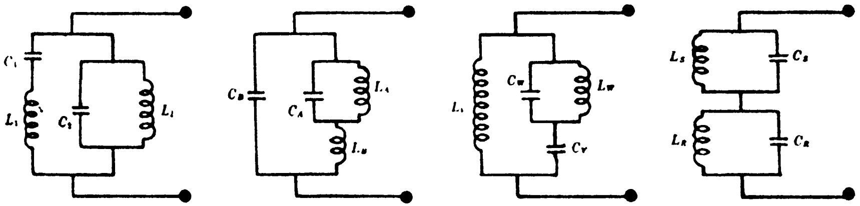


Fig. 12A Fig. 12B Fig. 12C Fig. 12D

FIG. 12. *One* type of four-element equivalent networks, each containing two inductances and two capacities.

Each of the meshes or networks shown in Fig. 12 are equivalent to each other from an impedance standpoint when the following relations have been satisfied:

$$L_1 = \frac{L_B(L_A + L_B)}{L_A} = L_W\left(1 + \frac{C_W}{C_V}\right)^2 = \frac{L_R L_S(L_R + L_S)(C_R + C_S)^2}{(L_R C_R - L_S C_S)^2} \quad (42)$$

$$L_2 = L_A + L_B = L_V = L_R + L_S \quad (43)$$

$$C_1 = \frac{C_A}{\left(1 + \frac{L_B}{L_A}\right)^2} = \frac{C_V^2}{C_V + C_W} = \frac{(L_R C_R - L_S C_S)^2}{(L_R + L_S)^2(C_R + C_S)} \quad (44)$$

$$C_2 = C_B = \frac{C_V C_W}{C_V + C_W} = \frac{C_R C_S}{C_R + C_S} \tag{45}$$

$$L_A = \frac{L_2^2}{L_1 + L_2} \tag{46}$$

$$L_B = \frac{L_1 L_2}{L_1 + L_2} \tag{47}$$

$$C_A = C_1\left(1 + \frac{L_1}{L_2}\right)^2 \tag{48}$$

$$C_B = C_2 \tag{49}$$

$$L_V = L_2 \tag{50}$$

$$L_W = \frac{L_1}{\left(1 + \frac{C_2}{C_1}\right)^2} \tag{51}$$

$$C_V = C_1 + C_2 \tag{52}$$

$$C_W = \frac{C_2}{C_1}(C_1 + C_2) \tag{53}$$

$$C_S = \frac{K + \sqrt{K^2 - 4L_2^2C_1C_2K}}{2L_2^2C_1} \tag{54}$$

in which $K \equiv (L_1C_1 + L_2C_1 + L_2C_2)^2 - 4L_1C_1L_2C_2$

$$C_R = \frac{C_SC_2}{C_S - C_2} \tag{55}$$

$$L_S = \frac{L_1C_1 + L_2C_1 + L_2C_2 - L_2C_R}{C_S - C_R} \tag{56}$$

$$L_R = L_2 - L_S \tag{57}$$

Note: In the special case in which $L_1C_1 = L_2C_2$, equations (54), (55), (56) and (57) reduce to:

$$C_S = \frac{C_1}{2}\left[\left(1 + \frac{4C_2}{C_1}\right) + \sqrt{1 + \frac{4C_2}{C_1}}\right] \tag{58}$$

$$C_R = \frac{C_1}{2}\left[\left(1 + \frac{4C_2}{C_1}\right) - \sqrt{1 + \frac{4C_2}{C_1}}\right] \tag{59}$$

$$L_S = \frac{L_2}{2}\left[1 + \frac{1}{\sqrt{1 + \frac{4L_1}{L_2}}}\right] \tag{60}$$

$$L_R = \frac{L_2}{2}\left[1 - \frac{1}{\sqrt{1 + \frac{4L_1}{L_2}}}\right] \tag{61}$$

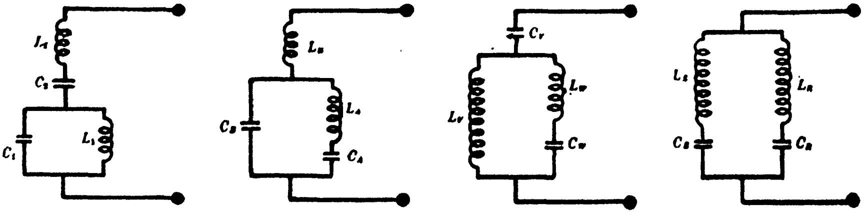


Fig. 13A Fig. 13B Fig. 13C Fig. 13D

FIG. 13. A *second* type of four-element equivalent networks, each containing two inductances and two capacities.

Each of the networks shown in Fig. 13 will have identically the same impedance, at all frequencies, when the following relations have been satisfied:

$$C_1 = \frac{C_B(C_A + C_B)}{C_A} = C_W\left(1 + \frac{L_W}{L_V}\right)^2 = \frac{C_R C_S(C_R + C_S)(L_R + L_S)^2}{(L_R C_R - L_S C_S)^2} \tag{62}$$

$$C_2 = C_A + C_B = C_V = C_R + C_S \tag{63}$$

$$L_1 = \frac{L_A}{\left(1 + \frac{C_B}{C_A}\right)^2} = \frac{L_V^2}{L_V + L_W} = \frac{(L_R C_R - L_S C_S)^2}{(C_R + C_S)^2(L_R + L_S)} \tag{64}$$

$$L_2 = L_B = \frac{L_V L_W}{L_V + L_W} = \frac{L_R L_S}{L_R + L_S} \tag{65}$$

$$C_A = \frac{C_2^2}{C_1 + C_2} \tag{66}$$

$$C_B = \frac{C_1 C_2}{C_1 + C_2} \tag{67}$$

$$L_A = L_1\left(1 + \frac{C_1}{C_2}\right)^2 \tag{68}$$

$$L_B = L_2 \tag{69}$$

$$C_V = C_2 \tag{70}$$

$$C_W = \frac{C_1}{\left(1 + \frac{L_2}{L_1}\right)^2} \tag{71}$$

$$L_V = L_1 + L_2 \tag{72}$$

$$L_W = \frac{L_2}{L_1}(L_1 + L_2) \tag{73}$$

$$L_S = \frac{K + \sqrt{K^2 - 4L_1L_2C_2^2K}}{2L_1C_2^2} \tag{74}$$

in which $K \equiv (L_1C_1 + L_1C_2 + L_2C_2)^2 - 4L_1C_1L_2C_2$

$$L_R = \frac{L_S L_2}{L_S - L_2} \tag{75}$$

$$C_S = \frac{L_1C_1 + L_1C_2 + L_2C_2 - L_RC_2}{L_S - L_R} \tag{76}$$

$$C_R = C_2 - C_S \tag{77}$$

Note: In the special case in which $L_1C_1 = L_2C_2$, equations (74), (75), (76) and (77) reduce to:

$$L_S = \frac{L_1}{2}\left[\left(1 + \frac{4L_2}{L_1}\right) + \sqrt{1 + \frac{4L_2}{L_1}}\right] \tag{78}$$

$$L_R = \frac{L_1}{2}\left[\left(1 + \frac{4L_2}{L_1}\right) - \sqrt{1 + \frac{4L_2}{L_1}}\right] \tag{79}$$

$$C_S = \frac{C_2}{2}\left[1 + \frac{1}{\sqrt{1 + \frac{4C_1}{C_2}}}\right] \tag{80}$$

$$C_R = \frac{C_2}{2}\left[1 - \frac{1}{\sqrt{1 + \frac{4C_1}{C_2}}}\right] \tag{81}$$

III. RELATIONS BETWEEN TWO-TERMINAL STRUCTURES COMPOSED OF RESISTANCES AND INDUCTANCES AND HAVING IDENTICALLY EQUIVALENT IMPEDANCE CHARACTERISTICS

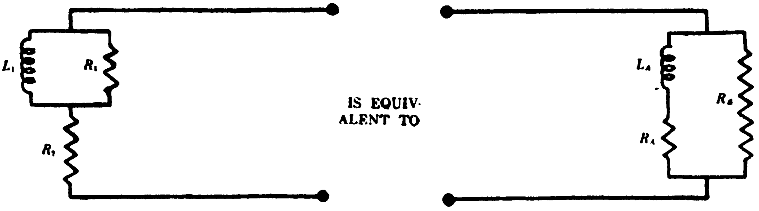


FIG. 14. Three-element equivalent networks, each containing one inductance and two resistances.

The two networks shown in Fig. 14 are equivalent when:

$$R_1 = \frac{R_B^2}{R_A + R_B} \tag{82}$$

$$R_2 = \frac{R_A R_B}{R_A + R_B} \tag{83}$$

$$L_1 = \frac{L_A}{\left(1 + \frac{R_A}{R_B}\right)^2} \tag{84}$$

$$R_A = R_2 + \frac{R_2^2}{R_1} \tag{85}$$

$$R_B = R_1 + R_2 \tag{86}$$

$$L_A = L_1\left(1 + \frac{R_2}{R_1}\right)^2 \tag{87}$$

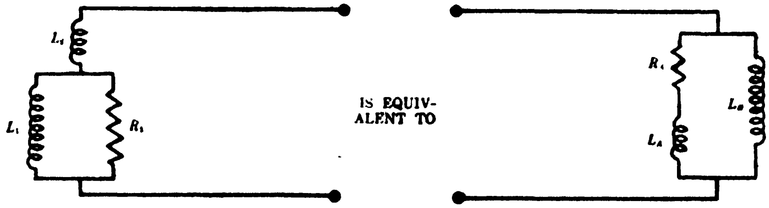


FIG. 15. Three-element equivalent networks, each containing one resistance and two inductances.

The two networks shown in Fig. 15 are equivalent when:

$$L_1 = \frac{L_B^2}{L_A + L_B} \tag{88}$$

$$L_2 = \frac{L_A L_B}{L_A + L_B} \tag{89}$$

$$R_1 = \frac{R_A}{\left(1 + \frac{L_A}{L_B}\right)^2} \tag{90}$$

$$L_B = L_1 + L_2 \tag{91}$$

$$L_A = \frac{L_2}{L_1}(L_1 + L_2) \tag{92}$$

$$R_A = R_1\left(1 + \frac{L_2}{L_1}\right)^2 \tag{93}$$

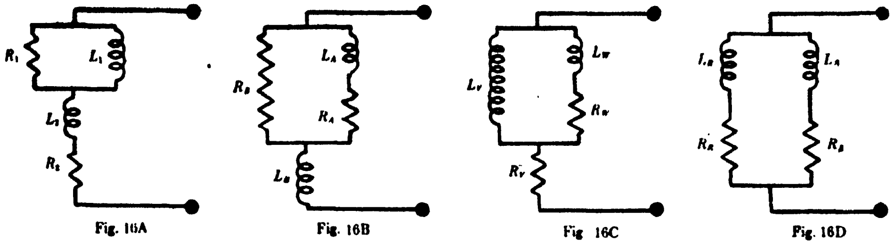


Fig. 16A Fig. 16B Fig 16C Fig. 16D

FIG. 16. *One* type of four-element equivalent networks, each containing two resistances and two inductances.

Each of the meshes shown in Fig. 16 will haye identically the same impedance, at all frequencies, provided the relations given below are satisfied:

$$R_1 = \frac{R_B^2}{R_A + R_B} = \frac{R_W}{\left(1 + \frac{L_W}{L_V}\right)^2} = \frac{(L_R R_S - L_S R_R)^2}{(L_R + L_S)^2 (R_R + R_S)} \tag{94}$$

$$R_2 = \frac{R_A R_B}{R_A + R_B} = R_V = \frac{R_R R_S}{R_R + R_S} \tag{94a}$$

$$L_2 = L_B = \frac{L_V L_W}{L_V + L_W} = \frac{L_R L_S}{L_R + L_S} \tag{95}$$

$$L_1 = \frac{L_A}{\left(1 + \frac{R_A}{R_B}\right)^2} = \frac{L_V^2}{L_V + L_W} = \frac{(L_R R_S - L_S R_R)^2}{(R_R + R_S)^2 (L_R + L_S)} \tag{96}$$

$$R_A = R_2 + \frac{R_2^2}{R_1} \tag{97}$$

$$R_B = R_1 + R_2 \tag{98}$$

$$L_B = L_2 \tag{99}$$

$$L_A = L_1\left(1 + \frac{R_2}{R_1}\right)^2 \tag{100}$$

$$R_V = R_2 \tag{101}$$

$$R_W = R_1\left(1 + \frac{L_2}{L_1}\right)^2 \tag{102}$$

$$L_V = L_1 + L_2 \tag{103}$$

$$L_W = \frac{L_2}{L_1}(L_1 + L_2) \tag{104}$$

$$L_S = \frac{R_2^2}{2L_1}\left[K + \sqrt{K^2 - \frac{4L_1L_2K}{R_2^2}}\right] \tag{105}$$

in which

$$K = \left(\frac{L_1}{R_1} + \frac{L_1}{R_2} + \frac{L_2}{R_2}\right)^2 - \frac{4L_1L_2}{R_1R_2}$$

$$L_R = \frac{L_S L_2}{L_S - L_2} \quad (106) \qquad R_S = \frac{L_S - L_R}{\frac{L_1}{R_1} + \frac{L_1}{R_2} + \frac{L_2}{R_2} - \frac{L_R}{R_2}} \quad (107)$$

$$R_R = \frac{R_2 R_S}{R_S - R_2} \quad (108)$$

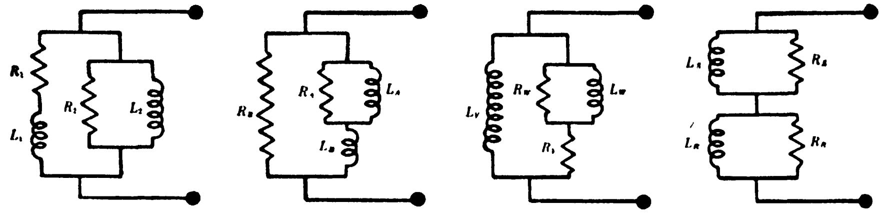


Fig. 17A Fig. 17B Fig. 17C Fig. 17D

FIG. 17. A *second* type of four-element equivalent networks, each containing two resistances and two inductances.

Likewise each of the meshes in Fig. 17 will have identically the same impedance, at all frequencies, provided the relations given below are satisfied:

$$L_1 = \frac{L_B}{L_A}(L_A + L_B) = L_W\left(1 + \frac{R_V}{R_W}\right)^2 = \frac{L_A L_S (L_R + L_S)(R_R + R_S)^2}{(L_R R_S - L_S R_R)^2} \quad (109)$$

$$L_2 = L_A + L_B = L_V = L_R + L_S \quad (110)$$

$$R_1 = R_A\left(1 + \frac{L_B}{L_A}\right)^2 = R_V + \frac{R_V^2}{R_W} = \frac{R_R R_S (R_R + R_S)(L_R + L_S)^2}{(L_R R_S - L_S R_R)^2} \quad (111)$$

$$R_2 = R_B = R_V + R_W = R_R + R_S \quad (112)$$

$$L_A = \frac{L_2^2}{L_1 + L_2} \quad (113) \qquad L_B = \frac{L_1 L_2}{L_1 + L_2} \quad (114)$$

$$R_A = \frac{R_1}{\left(1 + \frac{L_1}{L_2}\right)^2} \quad (115) \qquad R_B = R_2 \quad (116)$$

$$L_V = L_2 \quad (117) \qquad L_W = \frac{L_1}{\left(1 + \frac{R_1}{R_2}\right)^2} \quad (118)$$

$$R_V = \frac{R_1 R_2}{R_1 + R_2} \quad (119) \qquad R_W = \frac{R_2^2}{R_1 + R_2} \quad (120)$$

$$R_S = \frac{2L_2^2}{R_1} \cdot \frac{1}{K + \sqrt{K^2 - \frac{4L_2^2 K}{R_1 R_2}}} \quad (121)$$

in which

$$K = \left(\frac{L_1}{R_1} + \frac{L_2}{R_1} + \frac{L_2}{R_2}\right)^2 - \frac{4L_1L_2}{R_1R_2}$$

$$R_R = R_2 - R_S \tag{122}$$

$$L_S = \left(\frac{L_1}{R_1} + \frac{L_2}{R_1} + \frac{L_2}{R_2} - \frac{L_2}{R_R}\right)\frac{R_RR_S}{R_R - R_S} \tag{123}$$

$$L_R = L_2 - L_S \tag{124}$$

IV. RELATIONS BETWEEN TWO-TERMINAL STRUCTURES COMPOSED OF RESISTANCES AND CONDENSERS AND HAVING IDENTICALLY EQUIVALENT IMPEDANCE CHARACTERISTICS

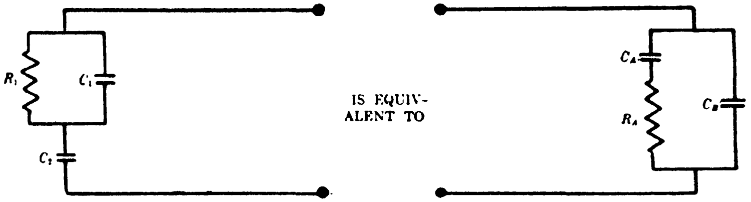


FIG. 18. Three-element equivalent networks, each containing one resistance and two capacities.

The two networks shown in Fig. 18 are equivalent when:

$$C_2 = C_A + C_B \tag{125}$$

$$C_1 = \frac{C_B}{C_A}(C_A + C_B) \tag{126}$$

$$R_1 = \frac{R_A}{\left(1 + \frac{C_B}{C_A}\right)^2} \tag{127}$$

$$C_A = \frac{C_2^2}{C_1 + C_2} \tag{128}$$

$$C_B = \frac{C_1C_2}{C_1 + C_2} \tag{129}$$

$$R_A = R_1\left(1 + \frac{C_1}{C_2}\right)^2 \tag{130}$$

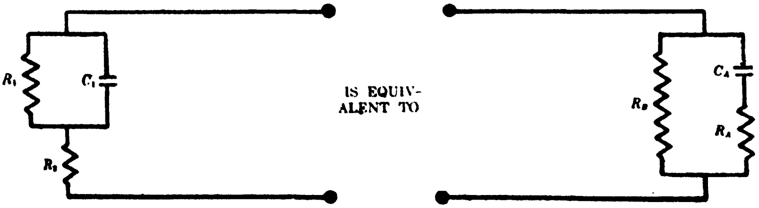


FIG. 19. Three-element equivalent networks, each containing one capacity and two resistances.

The two networks shown in Fig. 19 are equivalent when:

$$R_1 = \frac{R_B^2}{R_A + R_B} \qquad (131) \qquad\qquad R_2 = \frac{R_A R_B}{R_A + R_B} \qquad (132)$$

$$C_1 = C_A\left(1 + \frac{R_A}{R_B}\right)^2 \qquad (133) \qquad\qquad R_A = \frac{R_2}{R_1}(R_1 + R_2) \qquad (134)$$

$$R_B = R_1 + R_2 \qquad (135) \qquad\qquad C_A = \frac{C_1}{\left(1 + \dfrac{R_2}{R_1}\right)^2} \qquad (136)$$

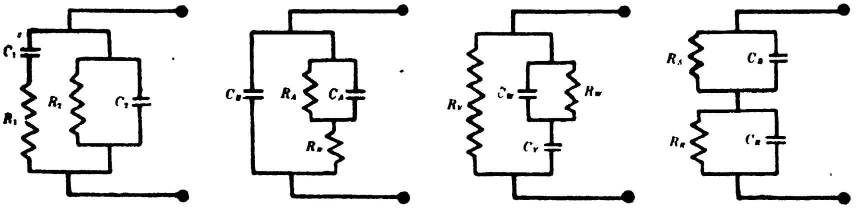


Fig. 20A Fig. 20B Fig. 20C Fig 20D

FIG. 20. *One* type of four-element equivalent networks, each containing two resistances and two capacities.

Each of the meshes of Fig. 20 will have identically the same impedance, at all frequencies, provided the relations given below are satisfied:

$$R_1 = \frac{R_B}{R_A}(R_A + R_B) = R_W\left(1 + \frac{C_W}{C_V}\right)^2 = \frac{R_R R_S (R_R + R_S)(C_R + C_S)^2}{(R_R C_R - R_S C_S)^2} \qquad (137)$$

$$R_2 = R_A + R_B = R_V = R_R + R_S \qquad (138)$$

$$C_1 = \frac{C_A}{\left(1 + \dfrac{R_B}{R_A}\right)^2} = \frac{C_V^2}{C_V + C_W} = \frac{(R_R C_R - R_S C_S)^2}{(R_R + R_S)^2 (C_R + C_S)} \qquad (139)$$

$$C_2 = C_B = \frac{C_V C_W}{C_V + C_W} = \frac{C_R C_S}{C_R + C_S} \qquad (140)$$

$$R_A = \frac{R_2^2}{R_1 + R_2} \qquad (141) \qquad\qquad R_B = \frac{R_1 R_2}{R_1 + R_2} \qquad (142)$$

$$C_A = C_1\left(1 + \frac{R_1}{R_2}\right)^2 \qquad (143) \qquad\qquad C_B = C_2 \qquad (144)$$

$$R_V = R_2 \qquad (145) \qquad\qquad R_W = \frac{R_1}{\left(1 + \dfrac{C_2}{C_1}\right)^2} \qquad (146)$$

$$C_V = C_1 + C_2 \qquad (147) \qquad\qquad C_W = \frac{C_2}{C_1}(C_1 + C_2) \qquad (148)$$

$$C_S = \frac{K + \sqrt{K^2 - 4R_2^2 C_1 C_2 K}}{2R_2^2 C_1} \qquad (149)$$

in which

$$K = (R_1C_1 + R_2C_1 + R_2C_2)^2 - 4R_1C_1R_2C_2$$

$$C_R = \frac{C_SC_2}{C_S - C_2} \tag{150}$$

$$R_S = \frac{R_1C_1 + R_2C_1 + R_2C_2 - R_2C_R}{C_S - C_R} \tag{151}$$

$$R_R = R_2 - R_S \tag{152}$$

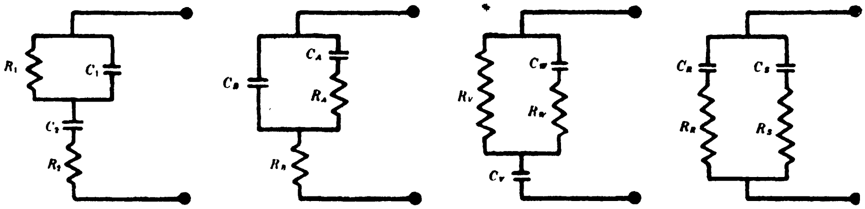


Fig 21A Fig 21B Fig 21C Fig 21D

FIG. 21. A *second* type of four-element equivalent networks, each containing two resistances and two capacities.

Each of the meshes of Fig. 21 will have identically the same impedance, at all frequencies, provided the relations given below are satisfied:

$$C_1 = \frac{C_B}{C_A}(C_A + C_B) = C_W\left(1 + \frac{R_W}{R_V}\right)^2 = \frac{C_RC_S(C_R + C_S)(R_R + R_S)^2}{(R_RC_R - L_SC_S)^2} \tag{153}$$

$$C_2 = C_A + C_B = C_V = C_R + C_S \tag{154}$$

$$R_1 = \frac{R_A}{\left(1 + \frac{C_B}{C_A}\right)^2} = \frac{R_V^2}{R_V + R_W} = \frac{(R_RC_R - R_SC_S)^2}{(C_R + C_S)^2(R_R + R_S)} \tag{155}$$

$$R_2 = R_B = \frac{R_VR_W}{R_V + R_W} = \frac{R_RR_S}{R_R + R_S} \tag{156}$$

$$C_A = \frac{C_2^2}{C_1 + C_2} \quad (157) \qquad C_B = \frac{C_1C_2}{C_1 + C_2} \quad (158)$$

$$R_A = R_1\left(1 + \frac{C_1}{C_2}\right)^2 \quad (159) \qquad R_B = R_2 \quad (160)$$

$$C_V = C_2 \quad (161) \qquad C_W = \frac{C_1}{\left(1 + \frac{R_2}{R_1}\right)^2} \quad (162)$$

$$R_V = R_1 + R_2 \quad (163) \qquad R_W = \frac{R_2}{R_1}(R_1 + R_2) \quad (164)$$

$$R_S = \frac{K + \sqrt{K^2 - 4R_1R_2C_2^2K}}{2R_1C_2^2} \tag{165}$$

in which

$$K = (R_1C_1 + R_1C_2 + R_2C_2)^2 - 4R_1R_2C_1C_2$$

$$R_R = \frac{R_SR_2}{R_S - R_2} \tag{166}$$

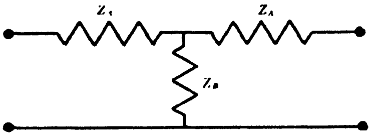
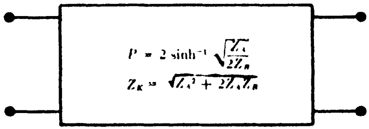
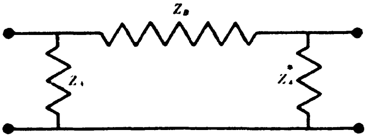
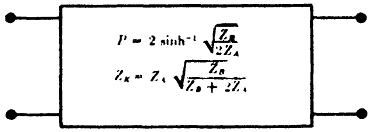
$$C_S = \frac{R_1C_1 + R_1C_2 + R_2C_2 - R_RC_2}{R_S - R_R} \tag{167}$$

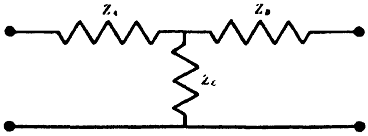
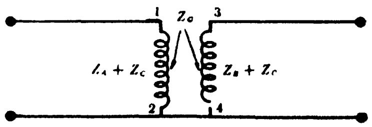
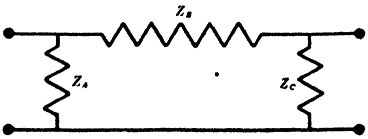
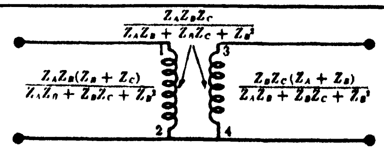
$$C_R = C_2 - C_S \tag{168}$$

V. RELATIONS BETWEEN EQUIVALENT FOUR-TERMINAL NETWORKS

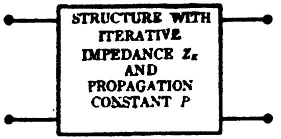
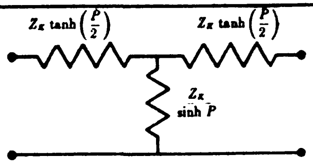
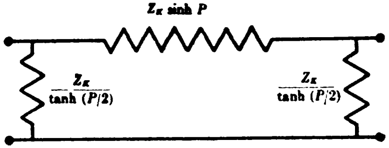
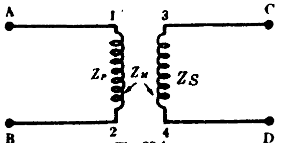
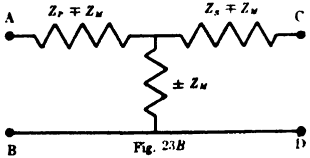
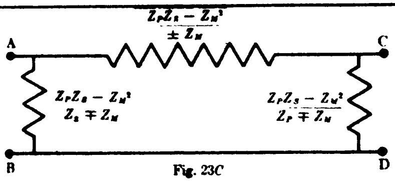
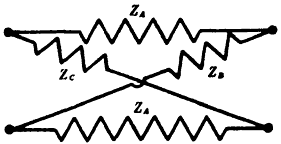
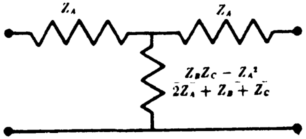
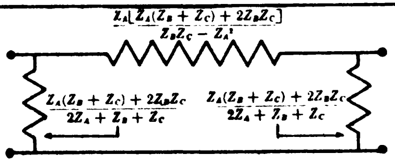
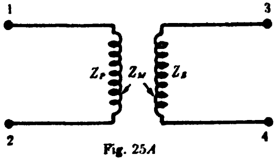
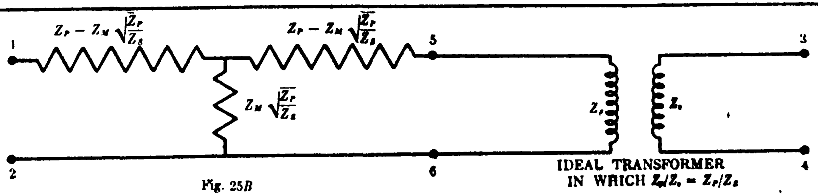
These are structures which, when connected between a source of power having only two terminals 1–2 (or 3–4) and an impedance also having only two terminals 3–4 (or 1–2), can be substituted for each other without affecting the current, voltage or power in either of the terminating circuits.*

(For mechanical reasons, the remaining figures in Appendix D do not appear in strictly numerical order).

STRUCTURE	EQUIVALENT SMOOTH LINE
 Fig. 31A	 $P = 2 \sinh^{-1} \sqrt{\frac{Z_A}{2Z_B}}$ $Z_K = \sqrt{Z_A^2 + 2Z_AZ_B}$ Fig. 31B
 Fig. 32A	 $P = 2 \sinh^{-1} \sqrt{\frac{Z_B}{2Z_A}}$ $Z_K = Z_A \sqrt{\frac{Z_B}{Z_B + 2Z_A}}$ Fig. 32B

STRUCTURE	EQUIVALENT 2 WINDING TRANSFORMER
 Fig. 33A	 Fig. 33B
 Fig. 34A	 $\frac{Z_AZ_BZ_C}{Z_AZ_B + Z_BZ_C + Z_B^2}$ 1 3 $\frac{Z_AZ_B(Z_B + Z_C)}{Z_AZ_B + Z_BZ_C + Z_B^2}$ $\frac{Z_BZ_C(Z_A + Z_B)}{Z_AZ_B + Z_BZ_C + Z_B^2}$ 2 4 Fig. 34B

Note: **In the circuits shown in Figs. 33B and 34B, the 1–2–3–4 connection of the transformers is assumed to be a series aiding connection.**

STRUCTURE	EQUIVALENT T NETWORKS	EQUIVALENT π NETWORKS
 STRUCTURE WITH ITERATIVE IMPEDANCE Z_K AND PROPAGATION CONSTANT P Fig. 22A	 $Z_K \tanh\left(\frac{P}{2}\right)$ $Z_K \tanh\left(\frac{P}{2}\right)$ $\frac{Z_K}{\sinh P}$ Fig. 22B	 $Z_K \sinh P$ $\frac{Z_K}{\tanh(P/2)}$ $\frac{Z_K}{\tanh(P/2)}$ Fig. 22C
 A B C D 1 2 3 4 Z_P Z_M ZS Fig. 23A	 A B C D $Z_P \mp Z_M$ $Z_S \mp Z_M$ $\pm Z_M$ Fig. 23B	 A B C D $\frac{Z_PZ_S - Z_M^2}{\pm Z_M}$ $\frac{Z_PZ_S - Z_M^2}{Z_S \mp Z_M}$ $\frac{Z_PZ_S - Z_M^2}{Z_P \mp Z_M}$ Fig. 23C
 Fig. 24A	 Fig. 24B	 $\frac{Z_A[Z_A(Z_B + Z_C) + 2Z_BZ_C]}{Z_BZ_C - Z_A^2}$ $\frac{Z_A(Z_B + Z_C) + 2Z_BZ_C}{2Z_A + Z_B + Z_C}$ $\frac{Z_A(Z_B + Z_C) + 2Z_BZ_C}{2Z_A + Z_B + Z_C}$ Fig. 24C
STRUCTURE	EQUIVALENT NETWORK—PROVIDED Z_P AND Z_S HAVE THE SAME PHASE ANGLES	
 Fig. 25A	 1 2 5 6 3 4 $Z_P - Z_M\sqrt{\frac{Z_P}{Z_S}}$ $Z_P - Z_M\sqrt{\frac{Z_P}{Z_S}}$ $Z_M\sqrt{\frac{Z_P}{Z_S}}$ Z_P Z_S IDEAL TRANSFORMER IN WHICH $Z_P/Z_S = Z_P/Z_S$ Fig. 25B	

<table>
<tr><th>STRUCTURE</th><th>EQUIVALENT T NETWORKS</th><th>EQUIVALENT π NETWORKS</th></tr>
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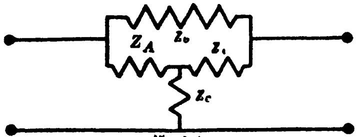
<td>

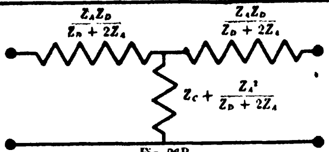
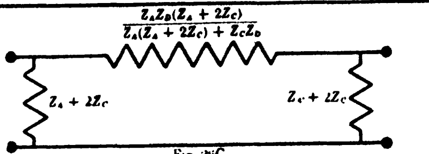
Fig. 26A</td>
<td>
$\frac{Z_A Z_D}{Z_D + 2Z_A}$
$\frac{Z_A Z_D}{Z_D + 2Z_A}$
$Z_C + \frac{Z_A^2}{Z_D + 2Z_A}$
Fig. 26B</td>
<td>

Fig. 26C</td>
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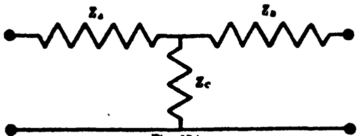
<td>

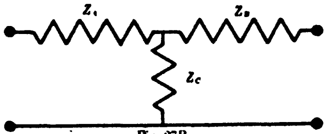
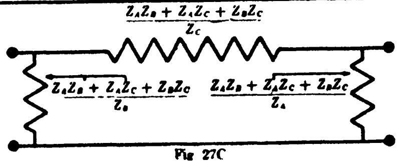
Fig. 27A</td>
<td>

Fig. 27B</td>
<td>
$\frac{Z_A Z_B + Z_A Z_C + Z_B Z_C}{Z_C}$
$\frac{Z_A Z_B + Z_A Z_C + Z_B Z_C}{Z_B}$
$\frac{Z_A Z_B + Z_A Z_C + Z_B Z_C}{Z_A}$
Fig 27C</td>
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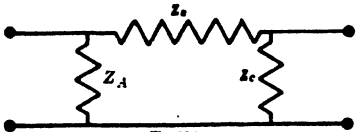
<td>

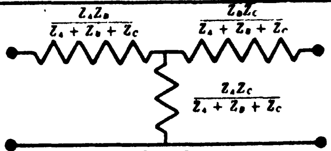
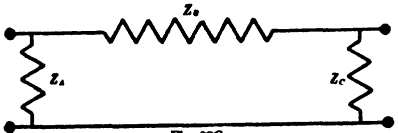
Fig. 28A</td>
<td>
$\frac{Z_A Z_B}{Z_A + Z_B + Z_C}$
$\frac{Z_B Z_C}{Z_A + Z_B + Z_C}$
$\frac{Z_A Z_C}{Z_A + Z_B + Z_C}$
Fig 28B</td>
<td>

Fig. 28C</td>
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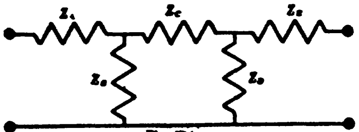
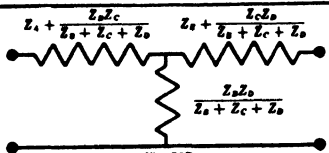
<td>

Fig. 29A</td>
<td>
$Z_A + \frac{Z_B Z_C}{Z_B + Z_C + Z_D}$
$Z_E + \frac{Z_C Z_D}{Z_B + Z_C + Z_D}$
$\frac{Z_B Z_D}{Z_B + Z_C + Z_D}$
Fig. 29B</td>
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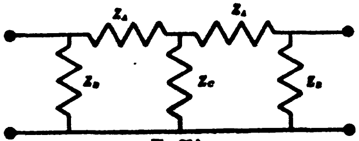
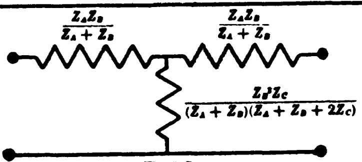
<td>

Fig. 30A</td>
<td>
$\frac{Z_A Z_B}{Z_A + Z_B}$
$\frac{Z_A Z_B}{Z_A + Z_B}$
$\frac{Z_B^2 Z_C}{(Z_A + Z_B)(Z_A + Z_B + 2Z_C)}$
Fig. 30B</td>
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APPENDIX E

FORMULAE FOR VARIOUS TYPES OF CIRCUITS *

I. Ladder Structure—Two Mesh.

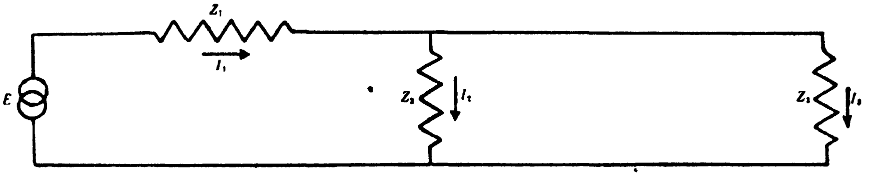

FIG. 1. Two-mesh ladder type of structure.

$$I_1 = \frac{E(Z_2 + Z_3)}{Z_1Z_2 + Z_1Z_3 + Z_2Z_3} \tag{1}$$

$$I_2 = \frac{EZ_3}{Z_1Z_2 + Z_1Z_3 + Z_2Z_3} \tag{2}$$

$$I_3 = \frac{EZ_2}{Z_1Z_2 + Z_1Z_3 + Z_2Z_3} \tag{3}$$

II. Ladder Structure—Three Mesh.

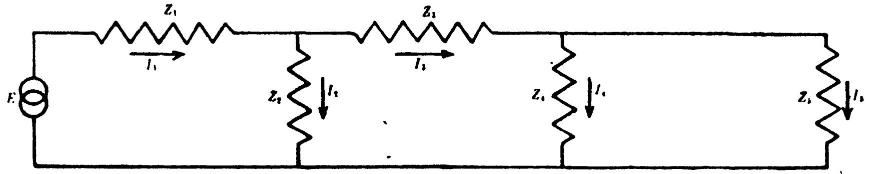

FIG. 2. Three-mesh ladder type of structure.

$$I_1 = \frac{E[(Z_2 + Z_3)(Z_4 + Z_5) + Z_4Z_5]}{H} \tag{4}$$

$$I_2 = \frac{E[Z_3(Z_4 + Z_5) + Z_4Z_5]}{H} \tag{5}$$

$$I_3 = \frac{EZ_2(Z_4 + Z_5)}{H} \tag{6}$$

$$I_4 = \frac{EZ_2Z_5}{H} \tag{7}$$

$$I_5 = \frac{EZ_2Z_4}{H} \tag{8}$$

---

* For circuit formulae involving separate-winding transformers or auto-transformers see Appendix G.

in all of which

$$H \equiv Z_1(Z_2 + Z_3)(Z_4 + Z_5) + Z_4Z_5(Z_1 + Z_2) + Z_2Z_3(Z_4 + Z_5) \quad (9)$$

III. Ladder Structure—Four Mesh.

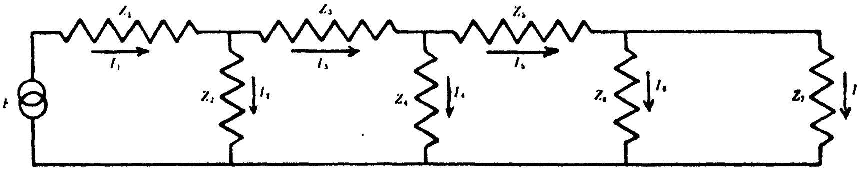

FIG. 3. Four-mesh ladder type of structure.

$$I_1 = \frac{E[\{Z_6+Z_7\}\{(Z_2+Z_3)(Z_4+Z_5)+Z_4Z_5\}+Z_6Z_7(Z_2+Z_3+Z_4)]}{H} \quad (10)$$

$$I_2 = \frac{E[\{Z_6+Z_7\}\{Z_5(Z_3+Z_4)+Z_3Z_4\}+Z_6Z_7(Z_3+Z_4)]}{H} \quad (11)$$

$$I_3 = \frac{EZ_2[(Z_4+Z_5)(Z_6+Z_7)+Z_6Z_7]}{H} \quad (12)$$

$$I_4 = \frac{EZ_2[Z_5(Z_6+Z_7)+Z_6Z_7]}{H} \quad (13)$$

$$I_5 = \frac{EZ_2Z_4(Z_6+Z_7)}{H} \quad (14)$$

$$I_6 = \frac{EZ_2Z_4Z_7}{H} \quad (15)$$

$$I_7 = \frac{EZ_2Z_4Z_6}{H} \quad (16)$$

in all of which

$$\begin{aligned} H \equiv {} & \{Z_6 + Z_7\}\{Z_1(Z_2 + Z_3)(Z_4 + Z_5) + Z_2Z_3(Z_4 + Z_5) \\ & + Z_4Z_5(Z_1 + Z_2)\} + Z_6Z_7\{(Z_1 + Z_2)(Z_3 + Z_4) + Z_1Z_2\} \end{aligned} \quad (17)$$

IV. Bridge Structure.

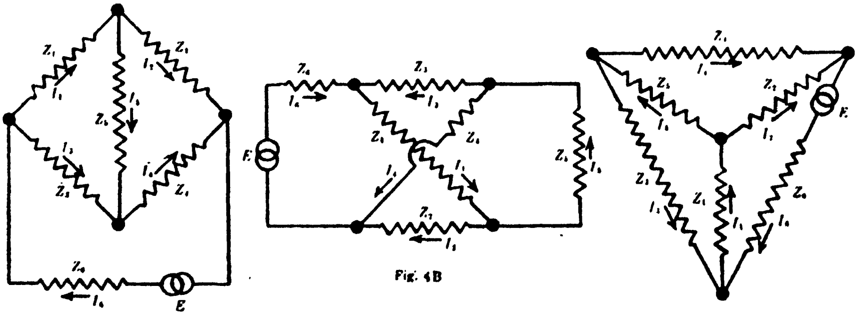

Fig. 4A

Fig. 4B

Fig. 4C

FIG. 4. Three different representations of a *bridge* type of structure.

Note: All of the circuits of Fig. 4 are identical—the particular method of representation being simply a matter of individual choice.

$$I_1 = \frac{E[Z_5(Z_3 + Z_4) + Z_3(Z_2 + Z_4)]}{H} \tag{18}$$

$$I_2 = \frac{E[Z_5(Z_3 + Z_4) + Z_4(Z_1 + Z_3)]}{H} \tag{19}$$

$$I_3 = \frac{E[Z_5(Z_1 + Z_2) + Z_1(Z_2 + Z_4)]}{H} \tag{20}$$

$$I_4 = \frac{E[Z_5(Z_1 + Z_2) + Z_2(Z_1 + Z_3)]}{H} \tag{21}$$

$$I_5 = \frac{E[Z_2Z_3 - Z_1Z_4]}{H} \tag{22}$$

$$I_6 = \frac{E[Z_5(Z_1 + Z_3) + (Z_2 + Z_4)(Z_1 + Z_3 + Z_5)]}{H} \tag{23}$$

in all of which

$$\begin{aligned} H \equiv\ & (Z_1 + Z_2)(Z_3Z_4 + Z_5Z_6) \\ & + (Z_3 + Z_4)(Z_1Z_2 + Z_5Z_6) + (Z_5 + Z_6)(Z_1Z_4 + Z_2Z_3) \\ & + Z_5(Z_1Z_3 + Z_2Z_4) + Z_6(Z_1Z_2 + Z_3Z_4) \end{aligned} \tag{24}$$

V. Bridged-T Structure.

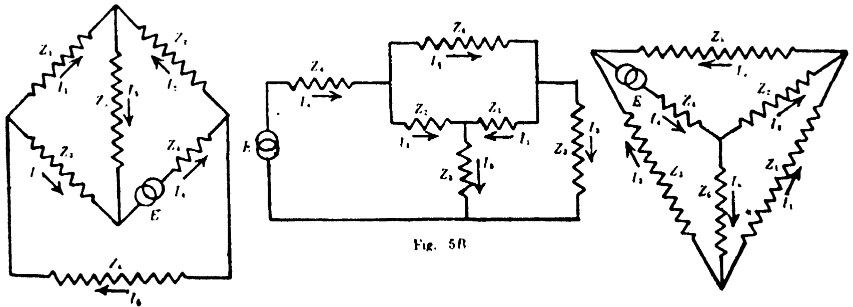

FIG. 5. Three different representations of a *bridged-T* type of structure.

Note: All of the circuits of Fig. 5 are identical—the particular method of representation being simply a matter of individual choice.

$$I_1 = \frac{E[Z_2Z_3 - Z_5Z_6]}{H} \tag{25}$$

$$I_2 = \frac{E[Z_1(Z_3 + Z_6) + Z_6(Z_3 + Z_5)]}{H} \tag{26}$$

$$I_3 = \frac{E[Z_1(Z_2 + Z_5) + Z_5(Z_2 + Z_6)]}{H} \qquad (27)$$

$$I_4 = \frac{E[Z_1(Z_2 + Z_6) + (Z_3 + Z_5)(Z_1 + Z_2 + Z_6)]}{H} \qquad (28)$$

$$I_5 = \frac{E[Z_1(Z_3 + Z_6) + Z_3(Z_2 + Z_6)]}{H} \qquad (29)$$

$$I_6 = \frac{E[Z_1(Z_2 + Z_5) + Z_2(Z_3 + Z_5)]}{H} \qquad (30)$$

in all of which, as in formula 24

$$\begin{aligned} H \equiv {} & (Z_1 + Z_2)(Z_3Z_4 + Z_5Z_6) \\ & + (Z_3 + Z_4)(Z_1Z_2 + Z_5Z_6) + (Z_5 + Z_6)(Z_1Z_4 + Z_2Z_3) \qquad (31) \\ & + Z_5(Z_1Z_3 + Z_2Z_4) + Z_6(Z_1Z_2 + Z_3Z_4) \end{aligned}$$

VI. Local Battery Substation Circuit.

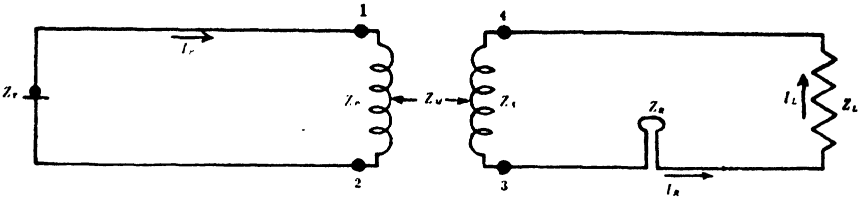

FIG. 6. Local battery type of substation circuit.

Note: 1–2–3–4 is assumed to give a series aiding connection of the induction coil.

Case 1: Assuming an e.m.f. E acting in series with Z_T, the various currents are:

$$I_R = I_L = \frac{EZ_M}{(Z_P + Z_T)(Z_S + Z_R + Z_L) - Z_M^2} \qquad (32)$$

$$I_P \equiv I_T = \frac{E(Z_S + Z_R + Z_L)}{(Z_P + Z_T)(Z_S + Z_R + Z_L) - Z_M^2} \qquad (33)$$

Case 2: Assuming an e.m.f. e acting in series with Z_L, the various currents are:

$$I_R = I_L = \frac{e(Z_P + Z_T)}{(Z_P + Z_T)(Z_S + Z_R + Z_L) - Z_M^2} \qquad (34)$$

$$I_P \equiv I_T = \frac{eZ_M}{(Z_P + Z_T)(Z_S + Z_R + Z_L) - Z_M^2} \qquad (35)$$

In an ideal circuit of the above type the power-distribution ratio is

$$Y = \frac{Z_S}{Z_P} \times \frac{Z_T}{Z_R} \tag{36}$$

The turns ratio on the induction coil is

$$\frac{N_S}{N_P} = \sqrt{\frac{Z_S}{Z_P}} = \sqrt{\frac{Z_L}{Z_T}\frac{Y}{Y+1}} \tag{37}$$

The impedance of the set is

$$Z_{set} = Z_R + \frac{Z_S}{Z_P} Z_T \tag{38}$$

The impedance of the receiver is

$$Z_R = \frac{Z_L}{Y+1} \tag{39}$$

VII. Common Battery Substation Circuit—General Case.

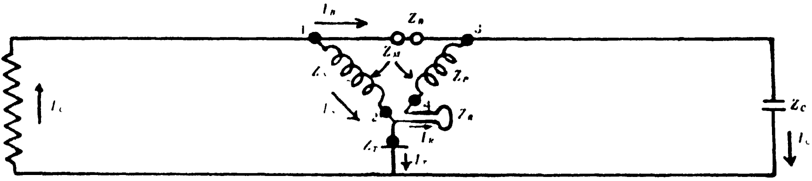

FIG. 7. Common battery type of substation circuit—general case.

Note: 1–2–4–3 is assumed to give a series aiding connection of the induction coil.

Case 1: Assuming an e.m.f. E acting in series with Z_T, the various currents are:

$$I_L = \frac{E[Z_C(Z_B + Z_R + Z_P + Z_S + 2Z_M) + Z_B(Z_R + Z_P + Z_M)]}{H} \tag{40}$$

$$I_R = \frac{-E[Z_B Z_L + (Z_S + Z_M)(Z_L + Z_C + Z_B)]}{H} \tag{41}$$

$$I_B = \frac{E[Z_C(Z_R + Z_P + Z_S + 2Z_M) - (Z_L + Z_C)(Z_R + Z_P + Z_M)]}{H} \tag{42}$$

$$I_T = \frac{\begin{aligned}&E[Z_B(Z_R + Z_P + Z_S + 2Z_M)\\ &\qquad + (Z_L + Z_C)(Z_B + Z_R + Z_P + Z_S + 2Z_M)]\end{aligned}}{H} \tag{43}$$

$$I_S = \frac{E[(Z_L + Z_C + Z_B)(Z_B + Z_R + Z_P + Z_M) - Z_B(Z_B + Z_L)]}{H} \tag{44}$$

$$I_C = \frac{-E[(Z_B + Z_L)(Z_S + Z_M) + Z_L(Z_B + Z_R + Z_P + Z_M)]}{H} \tag{45}$$

in all of which

$$\begin{aligned} H \equiv {} & [Z_C+Z_L][(Z_R+Z_C+Z_T+Z_P)(Z_R+Z_B+Z_P+Z_S+2Z_M) \\ & -(Z_R+Z_P+Z_M)^2]-[Z_R+Z_C+Z_P+Z_M] \\ & \times[Z_C(Z_R+Z_B+Z_P+Z_S+2Z_M)+Z_B(Z_R+Z_P+Z_M)] \\ & +[Z_R+Z_P+Z_S+2Z_M][Z_B(Z_R+Z_C+Z_T+Z_P) \\ & \qquad +Z_C(Z_R+Z_P+Z_M)] \end{aligned} \tag{46}$$

Case 2: Assuming an e.m.f. e acting in series with Z_L, the various currents are:

$$I_L=\frac{e[(Z_R+Z_B+Z_P+Z_S+2Z_M)(Z_R+Z_C+Z_T+Z_P)-(Z_R+Z_P+Z_M)^2]}{H} \tag{47}$$

$$I_R=\frac{-e[Z_B(Z_M-Z_T)+Z_C(Z_S+Z_M)]}{H} \tag{48}$$

$$I_B=\frac{e[(Z_R+Z_C+Z_T+Z_P)(Z_R+Z_P+Z_S+2Z_M)-(Z_R+Z_P+Z_M)(Z_R+Z_C+Z_P+Z_M)]}{H} \tag{49}$$

$$I_T=\frac{e[(Z_R+Z_C+Z_P+Z_M)(Z_R+Z_B+Z_P+Z_S+2Z_M)-(Z_R+Z_P+Z_M)(Z_R+Z_P+Z_S+2Z_M)]}{H} \tag{50}$$

$$I_S=\frac{e[Z_B(Z_R+Z_T+Z_P)+Z_C(Z_R+Z_B+Z_P+Z_M)]}{H} \tag{51}$$

$$I_C=\frac{e[(Z_S+Z_M)(Z_R+Z_T+Z_P)-(Z_M-Z_T)(Z_R+Z_B+Z_P+Z_M)]}{H} \tag{52}$$

in all of which H has the value given in (46).

VIII. Common Battery Substation Circuit—Ringer Omitted.

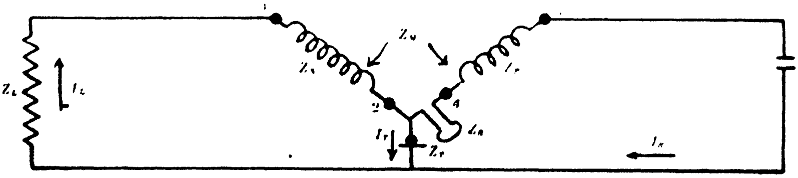

FIG. 8. Common battery type of substation circuit—ringer omitted.

Note: 1–2–4–3 gives a series aiding connection of the induction coil.

Case 1: Assuming an e.m.f. E acting in series with Z_T, the various currents are:

$$I_L=\frac{E[Z_R+Z_C+Z_P+Z_M]}{(Z_L+Z_T+Z_S)(Z_R+Z_C+Z_T+Z_P)-(Z_T-Z_M)^2} \tag{53}$$

$$I_R = \frac{-E[Z_L + Z_S + Z_M]}{(Z_L + Z_T + Z_S)(Z_R + Z_C + Z_T + Z_P) - (Z_T - Z_M)^2} \tag{54}$$

$$I_T = \frac{E[Z_R + Z_C + Z_L + Z_P + Z_S + 2Z_M]}{(Z_L + Z_T + Z_S)(Z_R + Z_C + Z_T + Z_P) - (Z_T - Z_M)^2} \tag{55}$$

Case 2: Assuming an e.m.f. e to be acting in series with Z_L, the various currents are:

$$I_L = \frac{e[Z_R + Z_C + Z_T + Z_P]}{(Z_L + Z_T + Z_S)(Z_R + Z_C + Z_T + Z_P) - (Z_T - Z_M)^2} \tag{56}$$

$$I_R = \frac{e[Z_T - Z_M]}{(Z_L + Z_T + Z_S)(Z_R + Z_C + Z_T + Z_P) - (Z_T - Z_M)^2} \tag{57}$$

$$I_T = \frac{e[Z_R + Z_C + Z_P + Z_M]}{(Z_L + Z_T + Z_S)(Z_R + Z_C + Z_T + Z_P) - (Z_T - Z_M)^2} \tag{58}$$

In an ideal substation circuit of the above type, the following relations hold:

The impedance of the set is:

$$Z_{\text{set}} = \frac{Z_S}{Z_P} Z_R + \left(\frac{\sqrt{Z_P} + \sqrt{Z_S}}{\sqrt{Z_P}}\right)^2 Z_T \tag{59}$$

The power-distribution ratio Y is (see Sect. 10.3)

$$Y = \left(\frac{\sqrt{Z_P} + \sqrt{Z_S}}{\sqrt{Z_S}}\right)^2 \times \frac{Z_T}{Z_R} \tag{60}$$

The turns ratio N_S/N_P of the induction coil is

$$\frac{N_S}{N_P} = \sqrt{\frac{Z_S}{Z_P}} = \pm\sqrt{\frac{Z_L}{Z_T} \times \frac{Y}{Y + 1}} - 1 \tag{61}$$

The receiver impedance is

$$Z_R = \frac{Z_L}{(Y + 1)\left(\pm\sqrt{\frac{Z_L}{Z_T}\frac{Y}{Y + 1}} - 1\right)^2} \tag{62}$$

IX. Local Battery Anti-Side-Tone Substation Circuit.

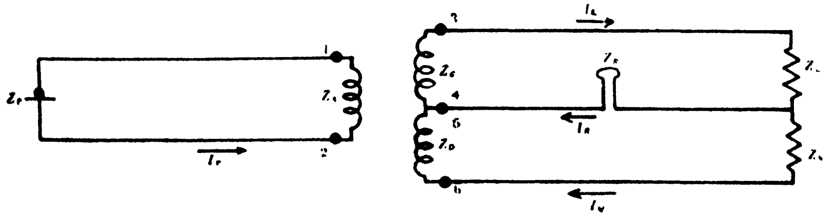

FIG. 9. Local battery anti-side-tone type of substation circuit.

Note: 1-2-3-4-5-6 gives a series aiding connection of the windings of the induction coil.

Case 1: Assuming an e.m.f. E acting in series with Z_T, the various currents are:

$$I_L = \frac{E[Z_{AC}(Z_D + Z_R + Z_N) - Z_{AD}(Z_{CD} - Z_R)]}{H} \tag{63}$$

$$I_R = \frac{E[Z_{AC}(Z_D + Z_{CD} + Z_N) - Z_{AD}(Z_C + Z_{CD} + Z_L)]}{H} \tag{64}$$

$$I_T = \frac{E[(Z_{CD} - Z_R)^2 - (Z_C + Z_R + Z_L)(Z_D + Z_R + Z_N)]}{H} \tag{65}$$

$$I_N = \frac{E[Z_{AD}(Z_C + Z_R + Z_L) - Z_{AC}(Z_{CD} - Z_R)]}{H} \tag{66}$$

in all of which

$$\begin{aligned} H \equiv [Z_A + Z_T][(Z_{CD} - Z_R)^2 - (Z_C + Z_R + Z_L)(Z_D + Z_R + Z_N)] \\ + Z_{AC}[Z_{AC}(Z_D + Z_R + Z_N) - Z_{AD}(Z_{CD} - Z_R)] \\ + Z_{AD}[Z_{AD}(Z_C + Z_R + Z_L) - Z_{AC}(Z_{CD} - Z_R)] \end{aligned} \tag{67}$$

and Z_{AC} is the mutual impedance between windings 1-2 and 3-4; Z_{CD} is the mutual impedance between windings 3-4 and 5-6, etc.

Case 2: Assuming an e.m.f. e acting in series with Z_L, the various currents are:

$$I_L = \frac{e[Z_{AD}^2 - (Z_A + Z_T)(Z_D + Z_R + Z_N)]}{H} \tag{68}$$

$$I_R = \frac{e[Z_{AD}(Z_{AC} + Z_{AD}) - (Z_A + Z_T)(Z_D + Z_{CD} + Z_N)]}{H} \tag{69}$$

$$I_T = \frac{e[Z_{AC}(Z_D + Z_R + Z_N) - Z_{AD}(Z_{CD} - Z_R)]}{H} \tag{70}$$

$$I_N = \frac{e[(Z_A + Z_T)(Z_{CD} - Z_R) - Z_{AC}Z_{AD}]}{H} \tag{71}$$

in which H has the value given in (67).

Assuming an ideal transformer, equations (63) to (71) become respectively:

With an e.m.f. E acting in Z_T:

$$I_L = \frac{E[Z_R(\sqrt{Z_A Z_C} + \sqrt{Z_A Z_D}) + Z_N \sqrt{Z_A Z_C}]}{H} \tag{72}$$

$$I_R = \frac{E[Z_N \sqrt{Z_A Z_C} - Z_L \sqrt{Z_A Z_D}]}{H} \tag{73}$$

$$I_T = \frac{-E[Z_R(\sqrt{Z_C} + \sqrt{Z_D})^2 + Z_N Z_C + Z_L Z_D]}{H} \tag{74}$$

$$I_N = \frac{E[Z_R(\sqrt{Z_A Z_C} + \sqrt{Z_A Z_D}) + Z_L\sqrt{Z_A Z_D}]}{H} \tag{75}$$

With an e.m.f. e acting in Z_L:

$$I_L = \frac{-e[Z_R Z_A + Z_N Z_A + Z_T Z_D]}{H} \tag{76}$$

$$I_R = \frac{-e[Z_T(Z_D + \sqrt{Z_C Z_D}) + Z_N Z_A]}{H} \tag{77}$$

$$I_T = \frac{e[Z_R(\sqrt{Z_A Z_C} + \sqrt{Z_A Z_D}) + Z_N\sqrt{Z_A Z_C}]}{H} \tag{78}$$

$$I_N = \frac{e[Z_T\sqrt{Z_C Z_D} - Z_R Z_A]}{H} \tag{79}$$

in all of which

$$H \equiv -[Z_A(Z_L Z_N + Z_L Z_R + Z_R Z_N) + Z_T(Z_C Z_N + Z_D Z_L) + Z_R Z_T(\sqrt{Z_C} + \sqrt{Z_D})^2]$$

In an ideal invariable substation circuit of the type shown in Fig. 9, the power-distribution ratio Y is

$$Y = \sqrt{\frac{Z_C}{Z_D}} = \frac{N_C}{N_D} \tag{80}$$

The impedance of the set is

$$Z_{set} = Z_R(Y + 1) \tag{81}$$

The relations between the various elements are given by

$$Z_L : Z_N : Z_T : Z_R = Y(Y + 1) : (Y + 1) : \frac{Z_A}{Z_D} : Y \tag{82}$$

X. Common Battery Anti-Side-Tone Substation Circuit.

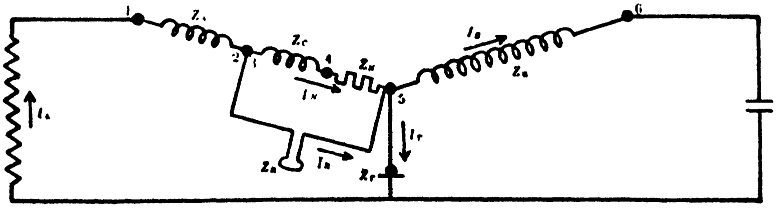

FIG. 10. Common battery anti-side-tone type of substation circuit.

Note: **1–2–3–4–5–6 is assumed to give a series aiding connection of the induction coil. Z_{AB} is the mutual impedance between winding 1–2 and winding 5–6, etc.**

Case 1: Assuming an e.m.f. E acting in series with Z_T, the various currents are:

$$I_L=\frac{E[Z_{BC}(Z_{AC}+Z_{BC}-Z_R)-(Z_C+Z_N+Z_R)(Z_B+Z_D+Z_{AB})]}{H} \tag{83}$$

$$I_R=\frac{\begin{aligned}E[Z_{BC}(Z_A+Z_B+2Z_{AB}+Z_{AC}+Z_{BC}+Z_D+Z_L)&\\ -(Z_C+Z_N+Z_{AC}+Z_{BC})(Z_B+Z_D+Z_{AB})]&\end{aligned}}{H} \tag{84}$$

$$I_T=\frac{\begin{aligned}E[(Z_{AC}+Z_{BC}-Z_R)(Z_C+Z_N+Z_{AC}+Z_{BC})&\\ -(Z_C+Z_N+Z_R)(Z_A+Z_B+2Z_{AB}+Z_{AC}+Z_{BC}+Z_D+Z_L)]&\end{aligned}}{H} \tag{85}$$

$$I_N=\frac{\begin{aligned}E[(Z_{AC}+Z_{BC}-Z_R)(Z_B+Z_D+Z_{AB})&\\ -Z_{BC}(Z_D+Z_L+Z_R+Z_A+Z_B+2Z_{AB})]&\end{aligned}}{H} \tag{86}$$

$$I_B=\frac{\begin{aligned}E[(Z_C+Z_N+Z_R)(Z_L+Z_A+Z_{AB}+Z_{AC}+Z_{BC})&\\ -(Z_C+Z_N+Z_{AC})(Z_{AC}+Z_{BC}-Z_R)]&\end{aligned}}{H} \tag{87}$$

in all of which

$$\begin{aligned}H\equiv{}&[Z_B+Z_D+Z_{AB}+Z_{BC}][(Z_C+Z_N+Z_R)(Z_B+Z_D+Z_{AB})\\ &-Z_{BC}(Z_{AC}+Z_{BC}-Z_R)]+[Z_C+Z_N+Z_{AC}+Z_{BC}]\\ &\times[(Z_B+Z_D+Z_T)(Z_{AC}+Z_{BC}-Z_R)-Z_{BC}(Z_B+Z_D+Z_{AB})]\\ &+[Z_A+Z_B+2Z_{AB}+Z_{AC}+Z_{BC}+Z_D+Z_L][Z_{BC}^2\\ &\qquad-(Z_C+Z_N+Z_R)(Z_B+Z_D+Z_T)]\end{aligned} \tag{88}$$

Case 2: Assuming an e.m.f. e acting in series with Z_L, the various currents are:

$$I_L=\frac{e[Z_{BC}^2-(Z_C+Z_N+Z_R)(Z_B+Z_D+Z_T)]}{H} \tag{89}$$

$$I_R=\frac{\begin{aligned}e[Z_{BC}(Z_B+Z_D+Z_{AB}+Z_{BC})&\\ -(Z_B+Z_D+Z_T)(Z_C+Z_N+Z_{AC}+Z_{BC})]&\end{aligned}}{H} \tag{90}$$

$$I_T=\frac{\begin{aligned}e[Z_{BC}(Z_C+Z_N+Z_{AC}+Z_{BC})&\\ -(Z_C+Z_N+Z_R)(Z_B+Z_D+Z_{AB}+Z_{BC})]&\end{aligned}}{H} \tag{91}$$

$$I_N=\frac{e[(Z_B+Z_D+Z_T)(Z_{AC}+Z_{BC}-Z_R)-Z_{BC}(Z_B+Z_D+Z_{AB})]}{H} \tag{92}$$

$$I_B=\frac{e[(Z_C+Z_N+Z_R)(Z_{AB}+Z_{BC}-Z_T)-Z_{BC}(Z_C+Z_N+Z_{AC})]}{H} \tag{93}$$

in all of which H has the value given in (88).

In an ideal substation circuit of the above type the power-distribution ratio is

$$Y = \frac{\sqrt{Z_A} + \sqrt{Z_B}}{\sqrt{Z_C}} \tag{94}$$

The turns ratios are given by

$$\frac{N_C}{N_B} \equiv \sqrt{\frac{Z_C}{Z_B}} = \pm\sqrt{\frac{Z_L}{Z_T} \times \frac{1}{Y(Y+1)}} \tag{95}$$

and

$$\frac{N_A}{N_C} \equiv \sqrt{\frac{Z_A}{Z_C}} = Y \mp \sqrt{\frac{Z_T}{Z_L} \times Y(Y+1)} \tag{96}$$

the upper of the alternative signs giving the most efficient circuit.

The relations between the various elements are given by

$$Z_L : Z_N : Z_R : Z_T = Y(Y+1) : (Y+1) : Y : \frac{Z_B}{Z_C} \tag{97}$$

XI. Attenuation Constant and Phase Constant Per Section of a Recurrent Ladder Type of Structure.

From formula (7) of Chapter XI

$$P = \cosh^{-1}\left[1 + \frac{Z_1}{2Z_2}\right] \tag{98}$$

Now if

$$U + jV \equiv \frac{Z_1}{4Z_2} \qquad \text{then} \qquad \sqrt{U^2 + V^2} = \left|\frac{Z_1}{4Z_2}\right| \tag{99}$$

Furthermore, if the phase angle of $Z_1/4Z_2$ is assumed to be ϕ, then

$$\cos\phi = \frac{U}{\sqrt{U^2 + V^2}} \qquad \text{or} \qquad U = \sqrt{U^2 + V^2}\cos\phi \tag{100}$$

Equation (98) now becomes

$$P = \cosh^{-1}\left[(1 + 2U) + j2V\right] \tag{101}$$

Referring to formula (80) of Appendix B, it is evident that equation (101) may be written

$$\begin{aligned} P = \cosh^{-1}&\left[\sqrt{V^2 + (1 + U)^2} + \sqrt{V^2 + U^2}\right] \\ &+ j\cos^{-1}\left[\sqrt{V^2 + (1 + U)^2} - \sqrt{V^2 + U^2}\right] \end{aligned} \tag{102}$$

If

$$k \equiv \left|\frac{Z_1}{4Z_2}\right| \equiv \sqrt{U^2 + V^2} \tag{103}$$

equations (100) and (102) give

$$\begin{aligned} P \equiv A + jB &= \cosh^{-1}\left[\pm\sqrt{k^2 + 2k\cos\phi + 1} + k\right] \\ &\qquad + j\cos^{-1}\left[\pm\sqrt{k^2 + 2k\cos\phi + 1} - k\right] \\ &= \sinh^{-1}\sqrt{2k\left[\pm\sqrt{k^2 + 2k\cos\phi + 1} + (k + \cos\phi)\right]} \\ &\qquad + j\sin^{-1}\sqrt{2k\left[\pm\sqrt{k^2 + 2k\cos\phi + 1} - (k + \cos\phi)\right]} \end{aligned} \qquad (104)$$

Since $\cos^2\frac{\phi}{2} = \frac{1}{2}(1 + \cos\phi)$, equation (104) may be written

$$\begin{aligned} P \equiv A + jB &= \cosh^{-1}\left[\pm\sqrt{(k-1)^2 + 4k\cos^2\frac{\phi}{2}} + k\right] \\ &\qquad + j\cos^{-1}\left[\pm\sqrt{(k-1)^2 + 4k\cos^2\frac{\phi}{2}} - k\right] \end{aligned} \qquad (105)$$

In equations (104) and (105) the upper of the alternative signs is to be used when k is greater than unity and the lower sign is to be used when k is less than unity.

If $\phi = 0°$:

$$A = \cosh^{-1}(2k + 1) = \sinh^{-1} 2k\sqrt{1 + \frac{1}{k}} \qquad (106)$$

If $\phi = 180°$:

$$A = \cosh^{-1}(2k - 1) = \sinh^{-1} 2k\sqrt{1 - \frac{1}{k}} \qquad (107)$$

If $k = 1$:

$$A = \cosh^{-1}\left(2\cos\frac{\phi}{2} + 1\right) \qquad (108)$$

If k is small compared with unity:

$$A \doteq 2\sqrt{k}\cos\frac{\phi}{2} \qquad (109)$$

If k is large compared with unity (say over 20):

$$A \doteq 2.3026\log_{10}(4k) \qquad (110)$$

If $\phi = 180°$ and if k is less than unity:

$$B = 2\sin^{-1}\sqrt{k} = \cos^{-1}(1 - 2k) = \sin^{-1}\sqrt{4k(1 - k)} \qquad (111)$$

If $k = 1$:

$$B = \cos^{-1}\left(2\cos\frac{\phi}{2} - 1\right) \qquad (112)$$

If k is small compared with unity:

$$B \doteqdot \sqrt{2k(1-k)(1-\cos\phi)} \tag{113}$$

If k approaches infinity:

$$B \doteqdot \phi \tag{114}$$

The above equations make it possible to determine the values of the attenuation constant (A) and the phase constant (B) per section of a recurrent structure of the ladder type (or its equivalent) provided one knows the value of one fourth the vector ratio of the series to the shunt impedances, or $Z_1/4Z_2 \equiv k/\phi$.

In order to facilitate the solution of problems where the value of $Z_1/4Z_2$ is known and where the value of the attenuation constant or the phase constant is desired, there is plotted in Fig. 11 (11A, 11B, 11C, 11D, 11E, 11F and 11G) a series of curves giving the relationship between k/ϕ and the attenuation constant A. Likewise, in Fig. 12 there is shown the relationship between k/ϕ and the phase constant B.

In using these curves, it should be pointed out that *the sign of the angle, ϕ (i.e. whether + or −), does not affect the value of the attenuation constant, A. On the other hand, the phase constant, B, always has the same sign as the angle ϕ.*

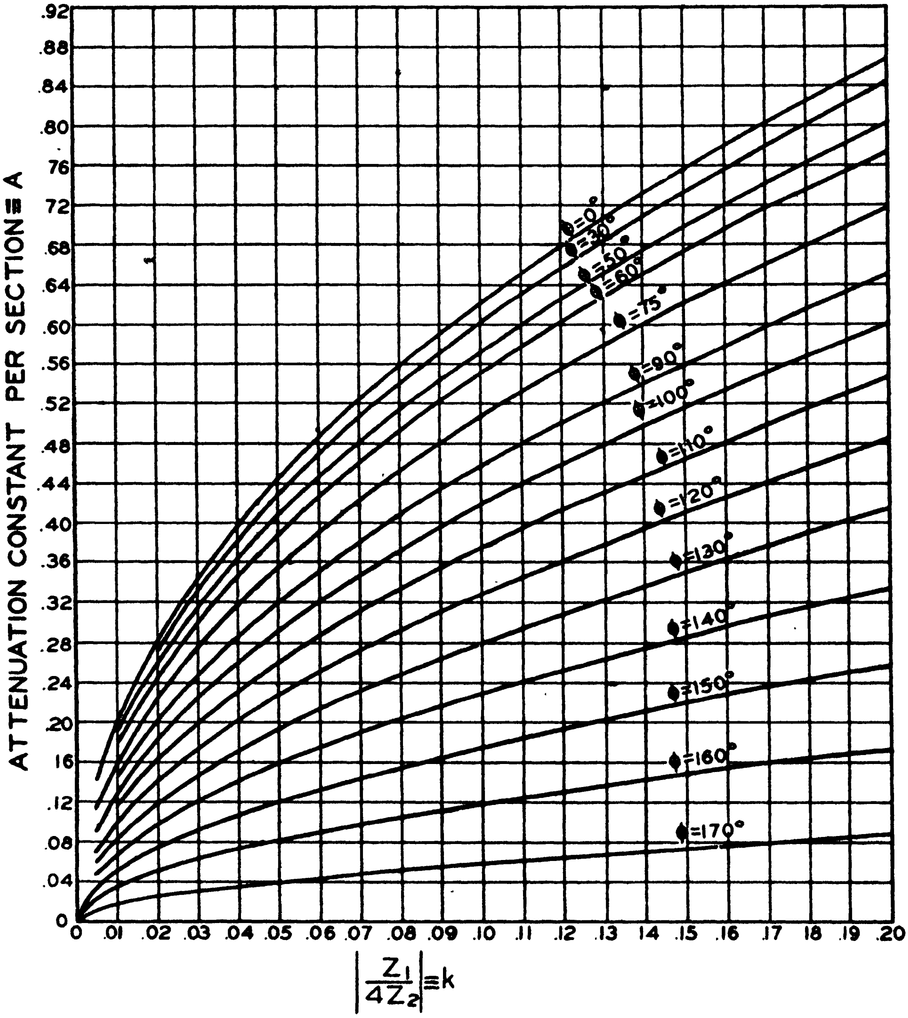

Fig. 11*A*. Curves showing the relation between the attenuation constant per section, *A*, of any ladder type of structure and one fourth the ratio of its series to its shunt impedances, $Z_1/4Z_2 = k\underline{/\phi}$. $k = 0$ to .2, $A = 0$ to .92.

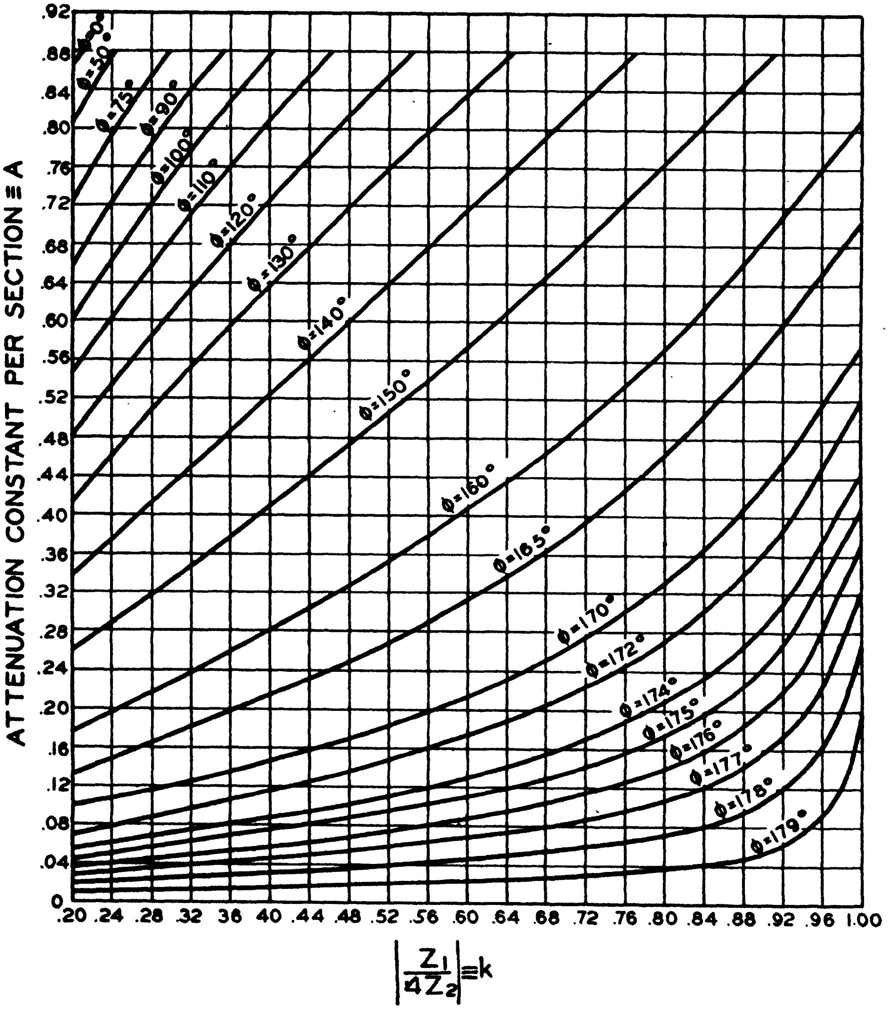

Fig. 11*B*. Curves showing the relation between the attenuation constant per section, *A*, of any ladder type of structure and one fourth the ratio of its series to its shunt impedances, $Z_1/4Z_2 \equiv k/\underline{\phi}$. $k = .2$ to 1.0, $A = 0$ to .92.

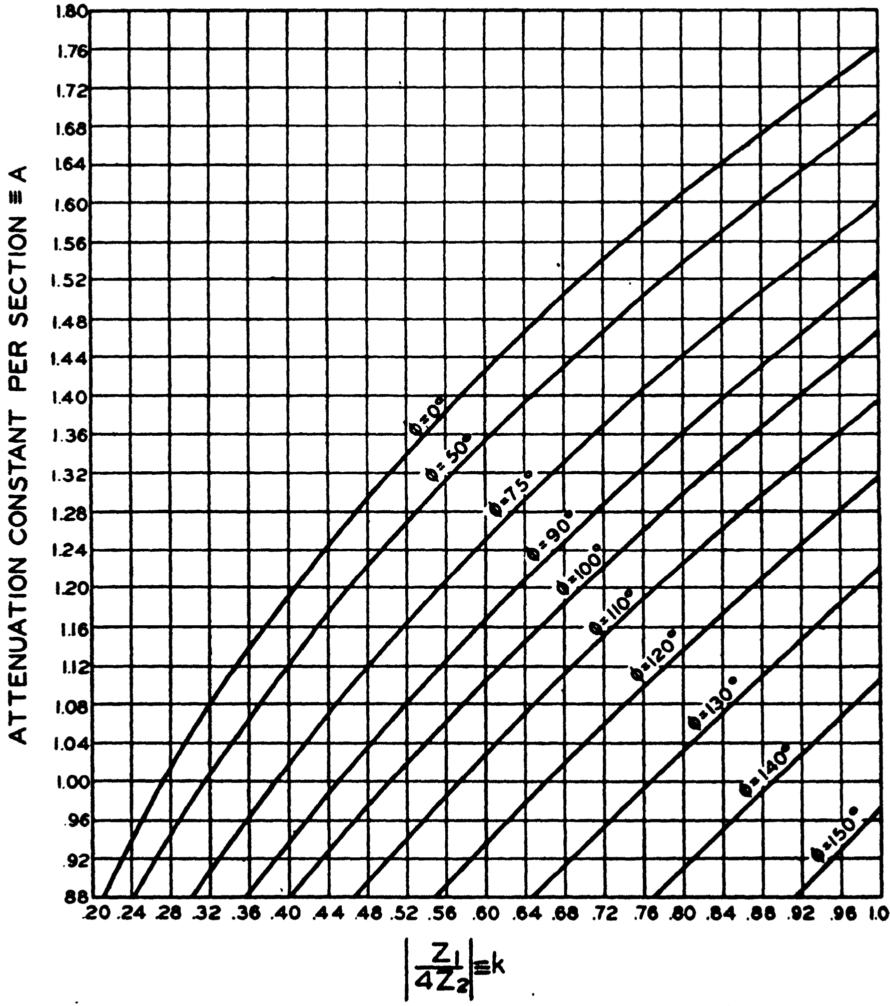

Fig. 11*C*. Curves showing the relation between the attenuation constant per section, *A*, of any ladder type of structure and one fourth the ratio of its series to its shunt impedances, $Z_1/4Z_2 = k\underline{/\phi}$. $k = .2$ to 1.0, $A = .88$ to 1.80.

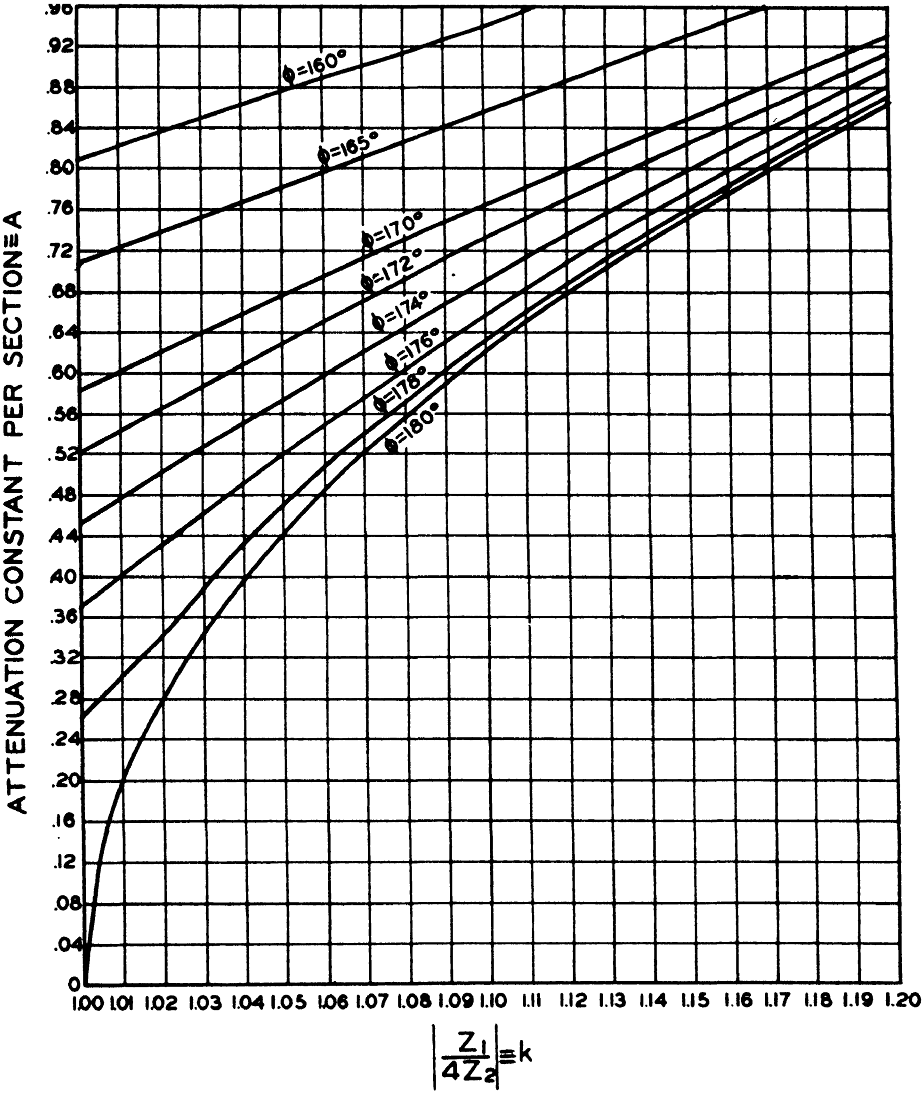

Fig. 11*D*. Curves showing the relation between the attenuation constant per section, *A*, of any ladder type of structure and one fourth the ratio of its series to its shunt impedances, $Z_1/4Z_2 = k\underline{/\phi}$. $k = 1.0$ to 1.2, $A = 0$ to $.96$.

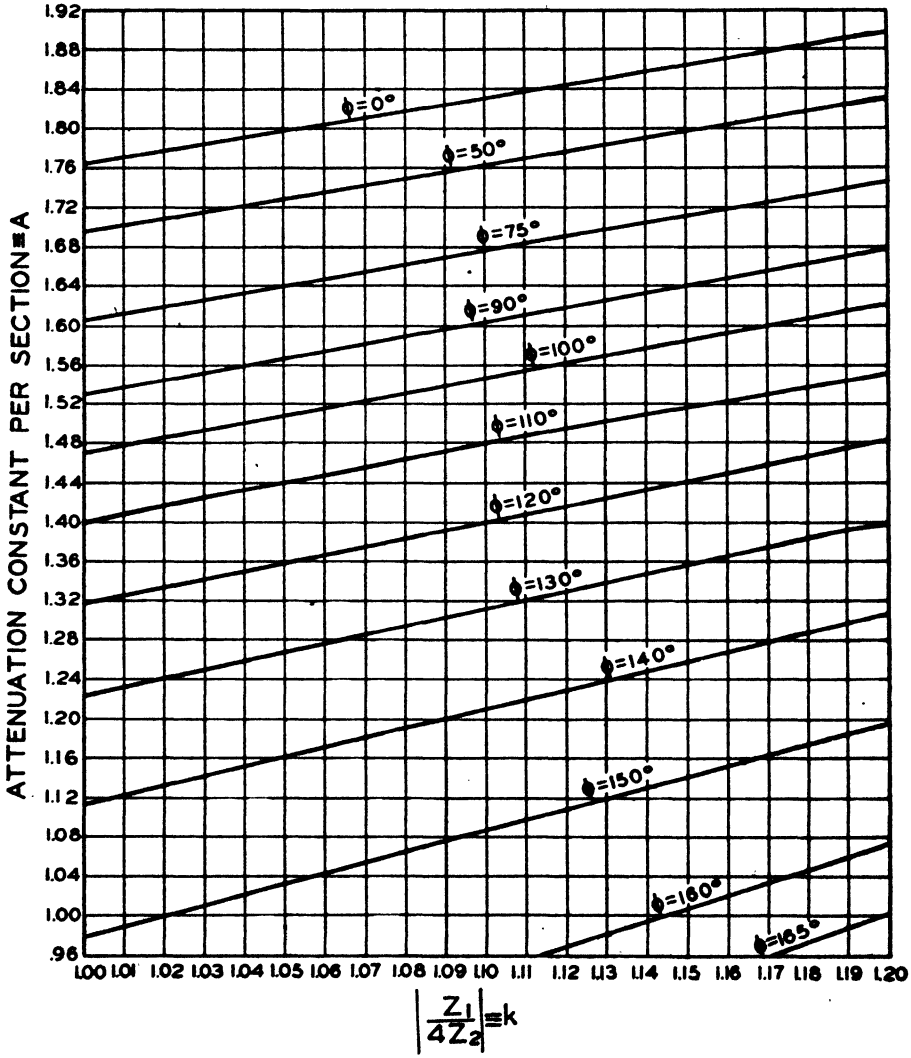

Fig. 11*E*. Curves showing the relation between the attenuation constant per section, *A*, of any ladder type of structure and one fourth the ratio of its series to its shunt impedances, $Z_1/4Z_2 = k\underline{/\phi}$. $k = 1.0$ to 1.2, $A = .96$ to 1.92.

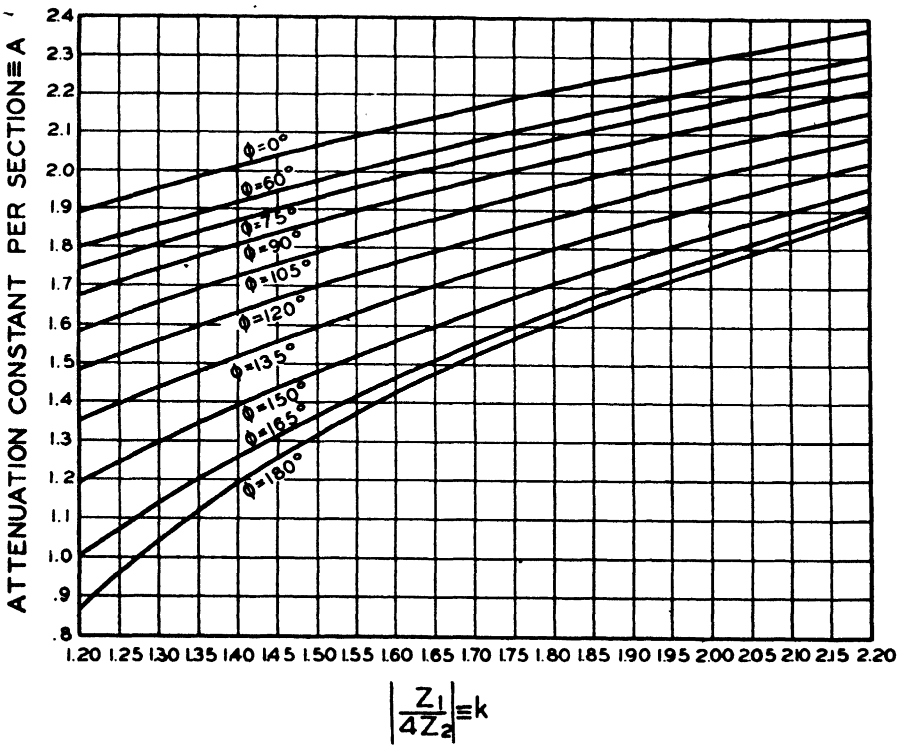

Fig. 11F. Curves showing the relation between the attenuation constant per section, A, of any ladder type of structure and one fourth the ratio of its series to its shunt impedances, $Z_1/4Z_2 \equiv k\underline{/\phi}$. $k = 1.2$ to 2.2. $A = .8$ to 2.4.

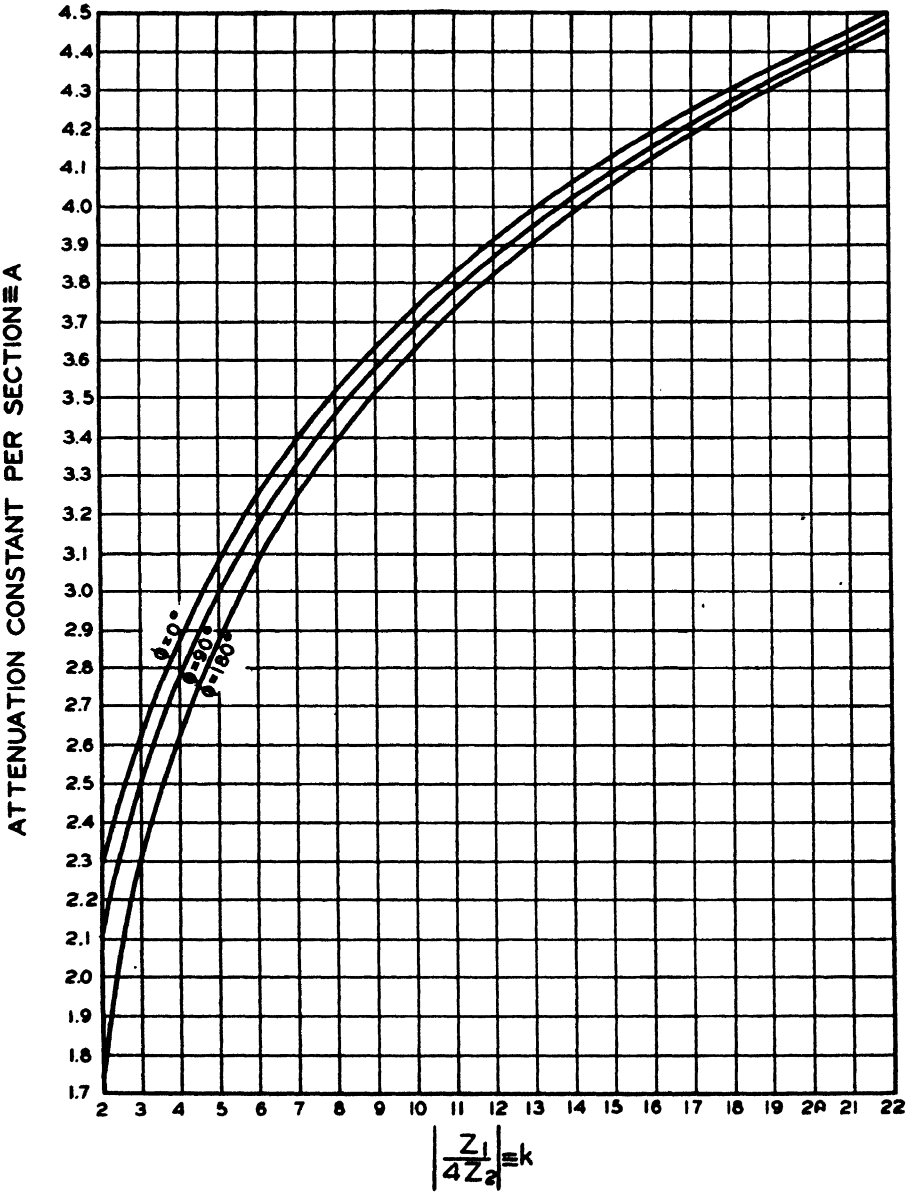

Fig. 11*G*. Curves showing the relation between the attenuation constant per section, *A*, of any ladder type of structure and one fourth the ratio of its series to its shunt impedances, $Z_1/4Z_2 = k\underline{/\phi}$. $k = 2$ to 22, $A = 1.7$ to 4.5.

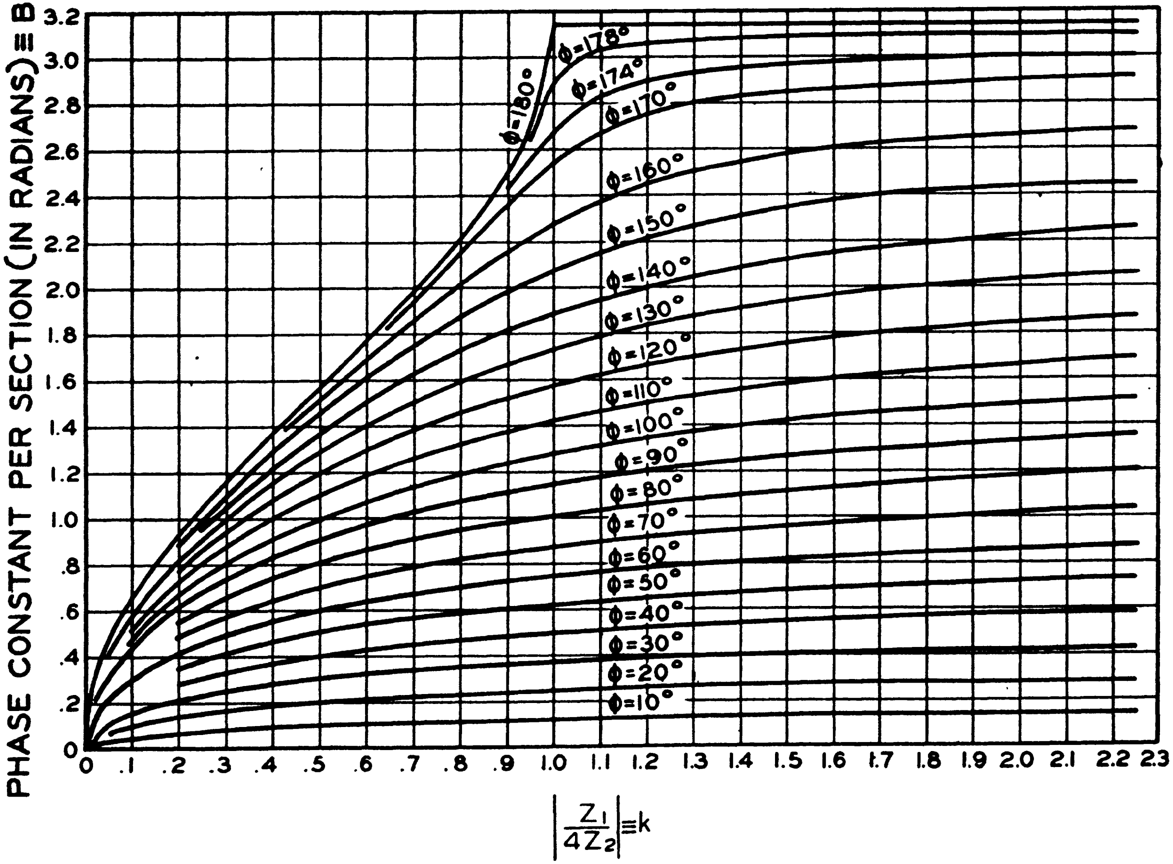

Fig. 12. Curves showing the relation between the phase constant per section, B, of any ladder type of structure and one fourth the ratio of its series to its shunt impedances, $Z_1/4Z_2 \equiv k/\underline{\phi}$.

APPENDIX F.

FORMULAE RELATING TO 2-, 3- AND 4-ELEMENT TWO-TERMINAL NETWORKS.

I. Formulae Relating to 2-Element Networks.

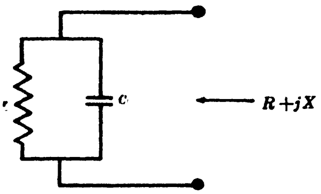

Fig. 1. A resistance, *r*, shunted by a capacity, *c*.

If, at any frequency $\omega/2\pi$, the impedance of the network shown in Fig. 1 is $R + jX$ ohms, then

$$R = \frac{r}{1+(rc\omega)^2} \quad (1) \qquad \text{and} \qquad X = -\frac{r^2 c\omega}{1+(rc\omega)^2} \quad (2)$$

Also, if we know that at ω_1, $R = R_1$ and that at ω_2, $R = R_2$, then

$$r = \frac{R_1 R_2(\omega_1^2 - \omega_2^2)}{R_1\omega_1^2 - R_2\omega_2^2} \quad (3) \qquad \text{and} \qquad c = \frac{\sqrt{\frac{r}{R_1} - 1}}{r\omega_1} \quad (4)$$

Similarly, if we know that at ω_1, $X = X_1$ and that at ω_2, $X = X_2$, then

$$c = \frac{X_1\omega_2 - X_2\omega_1}{X_1X_2(\omega_1^2 - \omega_2^2)} \quad (5) \qquad \text{and} \qquad r = \frac{1}{c\omega_1} \times \frac{1}{\sqrt{1 + \frac{1}{X_1 c\omega_1}}} \quad (6)$$

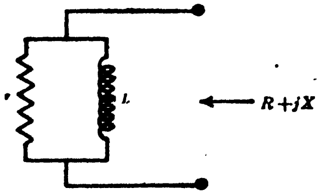

Fig. 2. A resistance, *r*, shunted by an inductance, *L*.

If, at any frequency $\omega/2\pi$, the impedance of the network shown in Fig. 2 is $R + jX$ ohms, then

$$R = \frac{r(L\omega)^2}{r^2 + (L\omega)^2} \quad (7) \qquad \text{and} \qquad X = \frac{r^2 L\omega}{r^2 + (L\omega)^2} \quad (8)$$

Also, if we know that at ω_1, $R = R_1$ and that at ω_2, $R = R_2$, then

$$r = \frac{R_1R_2(\omega_2^2 - \omega_1^2)}{R_1\omega_2^2 - R_2\omega_1^2} \quad (9) \qquad \text{and} \qquad L = \frac{r}{\omega_1} \times \frac{1}{\sqrt{\frac{r}{R_1} - 1}} \quad (10)$$

Similarly, if we know that at ω_1, $X = X_1$ and that at ω_2, $X = X_2$, then

$$L = \frac{X_1X_2(\omega_2^2 - \omega_1^2)}{\omega_1\omega_2(X_2\omega_2 - X_1\omega_1)} \quad (11) \qquad \text{and} \qquad r = \frac{L\omega_1}{\sqrt{\frac{L\omega_1}{X_1} - 1}} \quad (12)$$

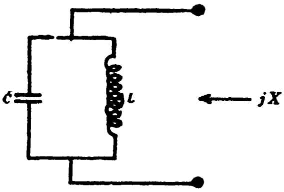

FIG. 3. An inductance, L, in parallel with a capacity, C

If, at any frequency $\omega/2\pi$, the impedance of the network shown in Fig. 3 is jX ohms, then

$$X = \frac{L\omega}{1 - LC\omega^2} \cdot \tag{13}$$

Also, if we know that at ω_1, $X = X_1$ and that at ω_2, $X = X_2$, then

$$L = \frac{X_1X_2(\omega_2^2 - \omega_1^2)}{\omega_1\omega_2(X_2\omega_2 - X_1\omega_1)} \quad (14) \qquad \text{and} \qquad C = \frac{X_2\omega_1 - X_1\omega_2}{X_1X_2(\omega_2^2 - \omega_1^2)} \quad (15)$$

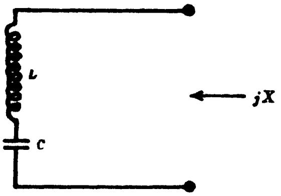

FIG. 4. An inductance, L, in series with a capacity, C.

If, at any frequency $\omega/2\pi$, the impedance of the network shown in Fig. 4 is jX ohms, then

$$X = \frac{LC\omega^2 - 1}{C\omega} \tag{16}$$

Also, if we know that at ω_1, $X = X_1$ and that at ω_2, $X = X_2$, then

$$C = \frac{\omega_2^2 - \omega_1^2}{\omega_1\omega_2(X_2\omega_1 - X_1\omega_2)} \quad (17) \qquad \text{and} \qquad L = \frac{X_2 + \frac{1}{C\omega_2}}{\omega_2} \quad (18)$$

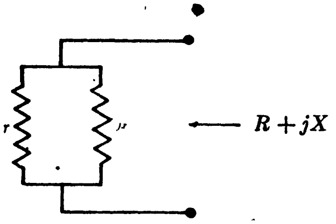

FIG. 5. A resistance, *r*, in parallel with a reactance, *x*.

If, at any frequency $\omega/2\pi$, the impedance of the network shown in Fig. 5 is $R + jX$ ohms, then

$$R = \frac{rx^2}{r^2 + x^2} \quad (19) \qquad \text{and} \qquad X = \frac{r^2 x}{r^2 + x^2} \quad (20)$$

$$r = \frac{x^2}{2R}\left[1 - \sqrt{1 - \left(\frac{2R}{x}\right)^2}\right] \quad (21) \qquad x = \frac{r}{\sqrt{\frac{r}{R} - 1}} \quad (22)$$

II. Formulae Relating to 3-Element Networks.

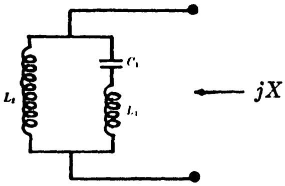

FIG. 6. An inductance, L_2, shunted by a resonant arm consisting of an inductance, L_1, in series with a capacity, C_1.

Given the network shown in Fig. 6, and the knowledge that its reactance at ω_1 is X_1, at ω_2 is X_2 and at ω_3 is X_3; to determine the values of L_1, C_1 and L_2.

We have

$$C_1 = \left[\frac{\omega_1}{X_1} - \frac{\omega_3}{X_3}\right]\left[\frac{(T\omega_1{}^2 - 1)(T\omega_3{}^2 - 1)}{\omega_1{}^2(T\omega_3{}^2 - 1) - \omega_3{}^2(T\omega_1{}^2 - 1)}\right] \quad (23)$$

$$L_1 = \frac{T}{C_1} \quad (24)$$

$$L_2 = \frac{1}{\dfrac{\omega_1}{X_1} - \dfrac{C_1\omega_1{}^2}{T\omega_1{}^2 - 1}} \quad (25)$$

the values of which may be calculated from the relations

$$V = \frac{1 - \dfrac{X_1}{X_2}\dfrac{\omega_2}{\omega_1}}{1 - \dfrac{X_1}{X_3}\dfrac{\omega_3}{\omega_1}} \quad (26)$$

and

$$T \equiv \frac{(\omega_3^2 - \omega_1^2)V - (\omega_2^2 - \omega_1^2)}{(\omega_3^2 - \omega_1^2)\omega_2^2 V - (\omega_2^2 - \omega_1^2)\omega_3^2} \tag{27}$$

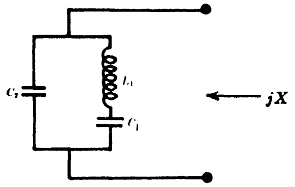

FIG. 7. A capacity, C_2, shunted by a resonant arm consisting of an inductance, L_1, in series with a capacity, C_1.

Similarly, if we are given the network shown in Fig. 7, and the knowledge that its reactance at ω_1 is X_1, at ω_2 is X_2, and at ω_3 is X_3, the values of L_1, C_1 and C_2 may be determined from the following equations.

$$C_1 = \frac{\dfrac{1}{X_1\omega_1} - \dfrac{1}{X_2\omega_2}}{\dfrac{1}{T\omega_1^2 - 1} - \dfrac{1}{T\omega_2^2 - 1}} \tag{28}$$

$$C_2 = \frac{C_1}{T\omega_2^2 - 1} - \frac{1}{X_2\omega_2} \tag{29}$$

$$L_1 = \frac{T}{C_1} \tag{30}$$

the values of which may be calculated from the relations

$$V \equiv \frac{1 - \dfrac{X_1}{X_2}\dfrac{\omega_1}{\omega_2}}{1 - \dfrac{X_1}{X_3}\dfrac{\omega_1}{\omega_3}} \tag{31}$$

and

$$T \equiv \frac{(\omega_2^2 - \omega_1^2) - V(\omega_3^2 - \omega_1^2)}{\omega_3^2(\omega_2^2 - \omega_1^2) - V\omega_2^2(\omega_3^2 - \omega_1^2)} \tag{32}$$

III. Formulae Relating to 4-Element Networks.

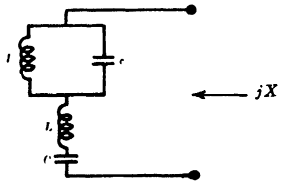

FIG. 8. A resonant arm—consisting of an inductance, L, in series with a capacity C—in series with an anti-resonant arm, consisting of an inductance, l, in parallel with a capacity, c.

Given the network represented in Fig. 8. Its impedance is pure reactance, X, and is given by

$$X = \left[\frac{LC\omega^2 - 1}{C\omega} + \frac{l\omega}{1 - lc\omega^2}\right] \tag{33}$$

If its reactance at ω_1 is X_1, at ω_2 is X_2, at ω_3 is X_3 and at ω_4 is X_4, the values of L, C, l and c may be determined as follows.

We have

$$(\omega_1^4)w + (X_1\omega_1^3)x + (\omega_1^2)y + (X_1\omega_1)z - 1 = 0 \tag{34}$$
$$(\omega_2^4)w + (X_2\omega_2^3)x + (\omega_2^2)y + (X_2\omega_2)z - 1 = 0 \tag{35}$$
$$(\omega_3^4)w + (X_3\omega_3^3)x + (\omega_3^2)y + (X_3\omega_3)z - 1 = 0 \tag{36}$$
$$(\omega_4^4)w + (X_4\omega_4^3)x + (\omega_4^2)y + (X_4\omega_4)z - 1 = 0 \tag{37}$$

Solving these four simultaneous equations (see equations 24, 25, 26 and 27 of Appendix A) for w, x, y and z, we then have

$$C = -z \tag{38}$$

$$L = -\frac{w}{x} \tag{39}$$

$$l = \frac{y}{C} + \frac{w}{LC^2} - L \tag{40}$$

$$c = \frac{x}{lC} \tag{41}$$

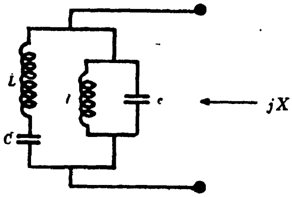

FIG. 9. A resonant arm—consisting of an inductance, L, in series with a capacity, C—in parallel with an anti-resonant arm, consisting of an inductance, l, in parallel with a capacity, c.

Similarly, suppose we have given the network of Fig. 9. Its impedance is pure reactance, X, and is given by

$$X = \left[\frac{1}{\dfrac{C\omega}{LC\omega^2 - 1} + \dfrac{1 - lc\omega^2}{l\omega}}\right] \tag{42}$$

If its reactance at ω_1 is X_1, at ω_2 is X_2, at ω_3 is X_3 and at ω_4 is X_4, the values of L, C, l and c may be determined as follows.

We have

$$(X_1\omega_1^4)w + (\omega_1^3)x + (X_1\omega_1^2)y + (\omega_1)z + X_1 = 0 \qquad (43)$$

$$(X_2\omega_2^4)w + (\omega_2^3)x + (X_2\omega_2^2)y + (\omega_2)z + X_2 = 0 \qquad (44)$$

$$(X_3\omega_3^4)w + (\omega_3^3)x + (X_3\omega_3^2)y + (\omega_3)z + X_3 = 0 \qquad (45)$$

$$(X_4\omega_4^4)w + (\omega_4^3)x + (X_4\omega_4^2)y + (\omega_4)z + X_4 = 0 \qquad (46)$$

Solving these four simultaneous equations (see equations 24, 25, 26 and 27 of Appendix A) for w, x, y and z, we then have

$$l = -z \qquad (47)$$

$$c = \frac{w}{x} \qquad (48)$$

$$C = -\left(c + \frac{y}{l} + \frac{w}{l^2 c}\right) \qquad (49)$$

$$L = \frac{x}{lC} \qquad (50)$$

If we are given the network shown in Fig. 10,

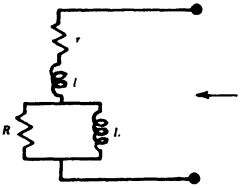

FIG. 10. A network consisting of one arm—composed of a resistance, r, in series with an inductance, l—in series with a second arm, composed of a resistance, R, in parallel with an inductance, L.

and the knowledge that its impedance, Z, at ω_1 is $R_1 + jL_1\omega_1$ and that at ω_2 it is $R_2 + jL_2\omega_2$, (in which if $\omega_2 > \omega_1$, then $R_2 > R_1$ and $L_1 > L_2$), the values of r, l, R and L may be determined as follows.

We have

$$R = \frac{R_2 - R_1}{\dfrac{\omega_2^2}{\left(\dfrac{R_2 - R_1}{L_1 - L_2}\right)^2 + \omega_2^2} - \dfrac{\omega_1^2}{\left(\dfrac{R_2 - R_1}{L_1 - L_2}\right)^2 + \omega_1^2}} \qquad (51)$$

$$L = R\frac{L_1 - L_2}{R_2 - R_1} \qquad (52)$$

$$r = R_2 - \frac{RL^2\omega_2^2}{R^2 + L^2\omega_2^2} \qquad (53)$$

$$l = L_2 - \frac{R^2L}{R^2 + L^2\omega_2^2} \qquad (54)$$

Similarly, if we are given the network of Fig. 11,

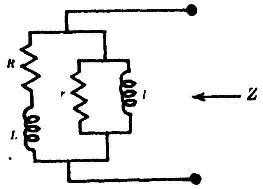

FIG. 11. A network consisting of one arm—composed of a resistance, R, in series with an inductance, L—in parallel with a second arm, composed of a resistance, r, in parallel with an inductance, l.

and the knowledge that its impedance, Z, at ω_1 is $R_1 + jL_1\omega_1$ and that at ω_2 it is $R_2 + jL_2\omega_2$, in which, if $\omega_2 > \omega_1$, then $R_2 > R_1$ and $L_1 > L_2$, the values of R, L, r and l may be determined as follows.

We have

$$R = \frac{\left(\frac{M_1 - M_2}{N_2 - N_1}\right)^2 \omega_2^2 - \left(\frac{M_1 - M_2}{N_2 - N_1}\right)^2 \omega_1^2}{\left[M_1 - M_2\right]\left[\left(\frac{M_1 - M_2}{N_2 - N_1}\right)^2 \omega_2^2 + 1\right]\left[\left(\frac{M_1 - M_2}{N_2 - N_1}\right)^2 \omega_1^2 + 1\right]} \tag{55}$$

$$L = \frac{M_1 - M_2}{N_2 - N_1} R \tag{56}$$

$$r = \frac{1}{M_1 - \frac{R}{R^2 + L^2\omega_1^2}} \tag{57}$$

$$l = \frac{1}{N_1 - \frac{L\omega_1^2}{R^2 + L^2\omega_1^2}} \tag{58}$$

in all of which

$$M_1 = \frac{R_1}{R_1^2 + L_1^2\omega_1^2} \quad (59) \qquad M_2 = \frac{R_2}{R_2^2 + L_2^2\omega_2^2} \quad (60)$$

$$N_1 = \frac{L_1\omega_1^2}{R_1^2 + L_1^2\omega_1^2} \quad (61) \qquad N_2 = \frac{L_2\omega_2^2}{R_2^2 + L_2^2\omega_2^2} \quad (62)$$

If we have given the network in Fig. 12,

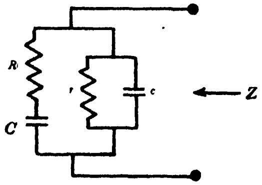

FIG. 12. A network consisting of one arm—composed of a resistance, R, in series with a capacity, C—in parallel with a second arm, composed of a resistance, r, in parallel with a capacity, c.

and the knowledge that its impedance, Z, at ω_1 is $R_1 - j/C_1\omega_1$ and that at ω_2 it is $R_2 - j/C_2\omega_2$, in which if $\omega_2 > \omega_1$, then $R_1 > R_2$ and $C_1 > C_2$, the values of R, C, r and c may be determined as follows.

We have

$$C = \frac{N_1 - N_2}{\dfrac{1}{1 + \left(\dfrac{N_1 - N_2}{M_2 - M_1}\right)^2 \omega_1^2} - \dfrac{1}{1 + \left(\dfrac{N_1 - N_2}{M_2 - M_1}\right)^2 \omega_2^2}} \tag{63}$$

$$R = \frac{1}{C}\left(\frac{N_1 - N_2}{M_2 - M_1}\right) \tag{64}$$

$$c = N_1 - \frac{C}{1 + \left(\dfrac{N_1 - N_2}{M_2 - M_1}\right)^2 \omega_1^2} \tag{65}$$

$$r = \frac{1}{M_1 - \dfrac{R}{R^2 + \dfrac{1}{C^2\omega_1^2}}} \tag{66}$$

in all of which

$$M_1 = \frac{R_1}{R_1^2 + \dfrac{1}{C_1^2\omega_1^2}} \quad (67) \qquad M_2 = \frac{R_2}{R_2^2 + \dfrac{1}{C_2^2\omega_2^2}} \quad (68)$$

$$N_1 = \frac{\dfrac{1}{C_1\omega_1^2}}{R_1^2 + \dfrac{1}{C_1^2\omega_1^2}} \quad (69) \qquad N_2 = \frac{\dfrac{1}{C_2\omega_2^2}}{R_2^2 + \dfrac{1}{C_2^2\omega_2^2}} \quad (70)$$

Similarly, if we have given the network represented in Fig. 13,

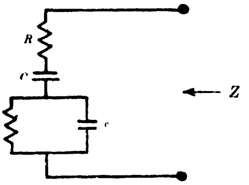

FIG. 13. A network consisting of one arm—composed of a resistance, R, in series with a capacity, C—in series with a second arm, composed of a resistance, r, in parallel with a capacity, c.

and the knowledge that its impedance, Z, at ω_1 is $R_1 - j/C_1\omega_1$ and that at ω_2 it is $R_2 - j/C_2\omega_2$, in which if $\omega_2 > \omega_1$, then $R_1 > R_2$ and $C_1 > C_2$, the values of R, C, r and c may be determined as follows.

We have

$$r = \frac{R_1 - R_2}{\dfrac{1}{1 + \left(\dfrac{R_1 - R_2}{\dfrac{1}{C_2} - \dfrac{1}{C_1}}\right)^2 \omega_1^2} - \dfrac{1}{1 + \left(\dfrac{R_1 - R_2}{\dfrac{1}{C_2} - \dfrac{1}{C_1}}\right)^2 \omega_2^2}} \tag{71}$$

$$c = \frac{R_1 - R_2}{r\left(\dfrac{1}{C_2} - \dfrac{1}{C_1}\right)} \tag{72}$$

$$R = R_1 - \frac{r}{1 + r^2c^2\omega_1^2} \tag{73}$$

$$C = \frac{1}{\dfrac{1}{C_1} - \dfrac{r^2c\omega_1^2}{1 + r^2c^2\omega_1^2}} \tag{74}$$

APPENDIX G

FORMULAE RELATING TO TRANSFORMERS

I. Transformer Having Two Separate Windings:

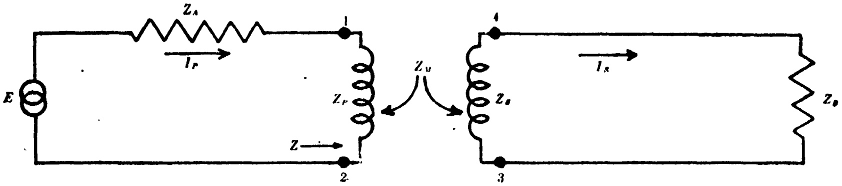

Fig. 1. Transformer having two separate windings.

$$I_P = \frac{E(Z_S + Z_B)}{(Z_P + Z_A)(Z_S + Z_B) - Z_M^2} \tag{1}$$

$$I_S = \frac{-EZ_M}{(Z_P + Z_A)(Z_S + Z_B) - Z_M^2} \tag{2}$$

Note: The above equations assume that 1–2–3–4 is the series aiding connection of the transformer.*

If the transformer in Fig. 1 is an ideal one, the currents are

$$I_P = \frac{EZ_S}{Z_BZ_P + Z_AZ_S} \tag{3}$$

and

$$I_S = \frac{-E\sqrt{Z_PZ_S}}{Z_BZ_P + Z_AZ_S} \tag{4}$$

From (3) the impedance "looking into" the 1–2 terminals is

$$Z = \frac{E}{I_P} - Z_A = \frac{Z_P}{Z_S} \times Z_B \tag{5}$$

If the transformer is an ideal one and of the best possible ratio (i.e., if $Z_S/Z_P = |Z_B/Z_A|$), the absolute magnitude of the received current is

$$|I_S| = \frac{E}{2\sqrt{|Z_AZ_B|}\cos\left(\frac{\theta - \phi}{2}\right)} \tag{6}$$

in which θ and ϕ are the phase angles respectively of the terminating impedances Z_A and Z_B.

---

* The effect of changing the direction of a winding of any transformer is to change the sign of any mutual impedance associated with that winding.

II. Two-Winding Auto-Transformer.

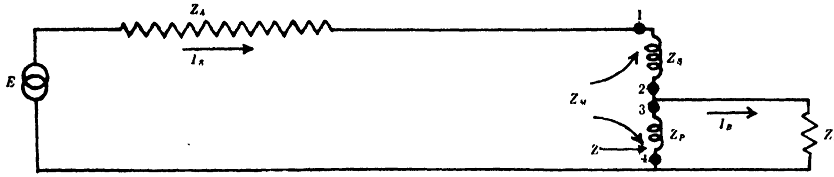

FIG. 2. Two-winding auto-transformer—with an e.m.f. applied across both windings.

$$I_S = \frac{E(Z_P + Z_B)}{(Z_P + Z_B)(Z_S + Z_M + Z_A) + (Z_P + Z_M)(Z_B - Z_M)} \tag{7}$$

$$I_B = \frac{E(Z_P + Z_M)}{(Z_P + Z_B)(Z_S + Z_M + Z_A) + (Z_P + Z_M)(Z_B - Z_M)} \tag{8}$$

Note: In the above equations, 1–2–3–4 is assumed to be the series aiding connection of the transformer.

If the transformer in Fig. 2 is an ideal one, the currents are

$$I_S = \frac{EZ_P}{Z_B(Z_P + Z_S + 2Z_M) + Z_AZ_P} \tag{9}$$

and

$$I_B = \frac{E(Z_P + Z_M)}{Z_B(Z_P + Z_S + 2Z_M) + Z_AZ_P} \tag{10}$$

From (9) the impedance "looking into" the 1–4 terminals of such an ideal auto-transformer is

$$Z = \frac{Z_P + Z_S + 2Z_M}{Z_P} \times Z_B \tag{11}$$

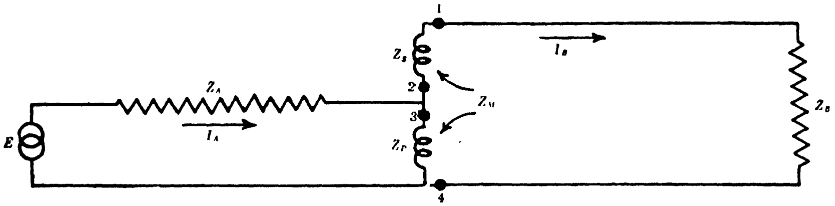

FIG. 3. Two-winding auto-transformer—with an e.m.f. applied across one winding.

$$I_B = \frac{E(Z_P + Z_M)}{(Z_P + Z_A)(Z_S + Z_P + 2Z_M + Z_B) - (Z_P + Z_M)^2} \tag{12}$$

$$I_A = \frac{E(Z_P + Z_S + 2Z_M + Z_B)}{(Z_P + Z_A)(Z_S + Z_P + 2Z_M + Z_B) - (Z_P + Z_M)^2} \tag{13}$$

Note: In the above equations, 1–2–3–4 is assumed to be the series aiding connection of the transformer.

III. Transformer Having Three Separate Windings:

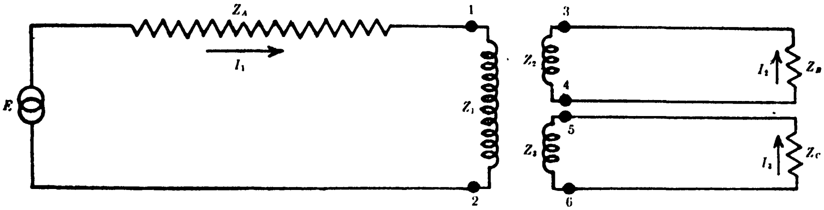

FIG. 4. Transformer having three separate windings.

$$I_1 = \frac{E[(Z_2 + Z_B)(Z_3 + Z_C) - Z_{23}^2]}{H} \tag{14}$$

$$I_2 = \frac{E[Z_{13}Z_{23} - Z_{12}(Z_3 + Z_C)]}{H} \tag{15}$$

$$I_3 = \frac{E[Z_{12}Z_{23} - Z_{13}(Z_2 + Z_B)]}{H} \tag{16}$$

Note: In the above equations, 1–2–3–4–5–6 is assumed to be the series aiding connection of the transformer. Z_{12} is the mutual impedance between Z_1 and Z_2, etc.

In each of the last three equations

$$\begin{aligned} H \equiv Z_{12}[Z_{13}Z_{23} - Z_{12}(Z_3 + Z_C)] + Z_{23}[Z_{12}Z_{13} - Z_{23}(Z_1 + Z_A)] \\ + [Z_2 + Z_B][(Z_1 + Z_A)(Z_3 + Z_C) - Z_{13}^2] \end{aligned} \tag{17}$$

In the case of an ideal transformer the above equations reduce to

$$I_1 = \frac{E(Z_2Z_C + Z_3Z_B)}{Z_A(Z_2Z_C + Z_3Z_B) + Z_1Z_BZ_C} \tag{18}$$

$$I_2 = \frac{-EZ_CZ_{12}}{Z_A(Z_2Z_C + Z_3Z_B) + Z_1Z_BZ_C} \tag{19}$$

$$I_3 = \frac{-EZ_BZ_{13}}{Z_A(Z_2Z_C + Z_3Z_B) + Z_1Z_BZ_C} \tag{20}$$

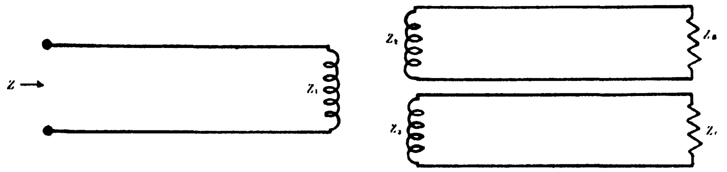

FIG. 5. Impedance "looking into" one winding of an ideal transformer having three separate windings.

From (18) it is noted that the impedance "looking into" the 1–2 winding of such an ideal transformer is

$$Z = \frac{E}{I_1} - Z_A = \frac{Z_1 Z_B Z_C}{Z_2 Z_C + Z_3 Z_B} \tag{21}$$

IV. Auto-Transformers Having Three Windings.

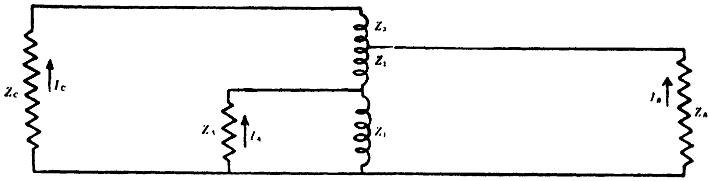

FIG. 6. Auto-transformer having three windings.

Note: The direction of the windings is such that if Z_A, Z_B and Z_C were infinite all three windings would be in a series aiding connection.

Case 1. With an e.m.f. E acting in series with Z_A:

$$I_A = \frac{\begin{array}{l} E[(Z_C+Z_1+Z_2+Z_3+2Z_{12}+2Z_{13}+2Z_{23})(Z_{13}+Z_{23}-Z_B) \\ \qquad -(Z_C+Z_3+Z_{13}+Z_{23})(Z_1+Z_2+2Z_{12}+Z_{13}+Z_{23})] \end{array}}{H} \tag{22}$$

$$I_B = \frac{\begin{array}{l} E[(Z_1+Z_{12}+Z_{13})(Z_C+Z_3+Z_{13}+Z_{23}) \\ \qquad - Z_{13}(Z_C+Z_1+Z_2+Z_3+2Z_{12}+2Z_{13}+2Z_{23})] \end{array}}{H} \tag{24}$$

$$I_C = \frac{\begin{array}{l} E[Z_{13}(Z_1+Z_2+2Z_{12}+Z_{13}+Z_{23}) \\ \qquad +(Z_B-Z_{13}-Z_{23})(Z_1+Z_{12}+Z_{13})] \end{array}}{H} \tag{25}$$

Case 2. With an e.m.f. E acting in series with Z_B:

$$I_A = \frac{\begin{array}{l} E[(Z_1+Z_{12}+Z_{13})(Z_C+Z_3+Z_{13}+Z_{23}) \\ \qquad -Z_{13}(Z_C+Z_1+Z_2+Z_3+2Z_{12}+2Z_{13}+2Z_{23})] \end{array}}{H} \tag{26}$$

$$I_B = \frac{\begin{array}{l} E[(Z_1+Z_{12}+Z_{13})^2 \\ \qquad -(Z_A+Z_1)(Z_C+Z_1+Z_2+Z_3+2Z_{12}+2Z_{13}+2Z_{23})] \end{array}}{H} \tag{27}$$

$$I_C = \frac{\begin{array}{l} E[(Z_A+Z_1)(Z_1+Z_2+2Z_{12}+Z_{13}+Z_{23}) \\ \qquad -(Z_1+Z_{12})(Z_1+Z_{12}+Z_{13})] \end{array}}{H} \tag{28}$$

Case 3. With an e.m.f. E acting in series with Z_C:

$$I_A = \frac{E[(Z_1+Z_{12}+Z_{13})(Z_2+Z_{12}+Z_B) - (Z_1+Z_{12})(Z_2+Z_{12}+Z_{23})]}{H} \tag{29}$$

$$I_B = \frac{E[(Z_A+Z_1)(Z_2+Z_{12}+Z_{23})+(Z_A-Z_{12})(Z_1+Z_{12}+Z_{13})]}{H} \tag{30}$$

$$I_C = \frac{E[(Z_1+Z_{12})(Z_{12}-Z_A)-(Z_A+Z_1)(Z_2+Z_{12}+Z_B)]}{H} \tag{31}$$

In all of the preceding 9 equations:

$$\begin{aligned} H \equiv {} & [Z_1+Z_{12}][(Z_C+Z_3+Z_{13}+Z_{23})(Z_{12}-Z_A)-Z_{13}(Z_2+Z_{12}+Z_{23})] \\ & +[Z_{13}+Z_{23}-Z_B][(Z_A+Z_1)(Z_2+Z_{12}+Z_{23}) \\ & \qquad +(Z_A-Z_{12})(Z_1+Z_{12}+Z_{13})] \\ & +[Z_2+Z_{12}+Z_B][Z_{13}(Z_1+Z_{12}+Z_{13}) \\ & \qquad -(Z_A+Z_1)(Z_C+Z_3+Z_{13}+Z_{23})] \end{aligned} \tag{32}$$

Assuming the auto-transformers, just considered, to be ideal, the impedances as determined at various terminals are as indicated below:

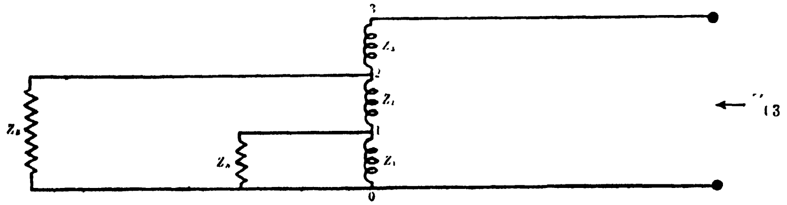

FIG. 7A. Impedance "looking into" the 0–3 terminals of an ideal three-winding auto-transformer.

$$Z_{03} = \frac{Z_A Z_B(\sqrt{Z_1}+\sqrt{Z_2}+\sqrt{Z_3})^2}{Z_B Z_1 + Z_A(\sqrt{Z_1}+\sqrt{Z_2})^2} \tag{33}$$

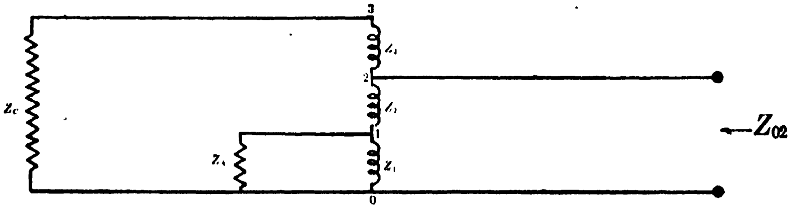

Fig. 7B. Impedance "looking into" the 0–2 terminals of an ideal three-winding auto-transformer.

$$Z_{02} = \frac{Z_A Z_C(\sqrt{Z_1}+\sqrt{Z_2})^2}{Z_C Z_1 + Z_A(\sqrt{Z_1}+\sqrt{Z_2}+\sqrt{Z_3})^2} \tag{34}$$

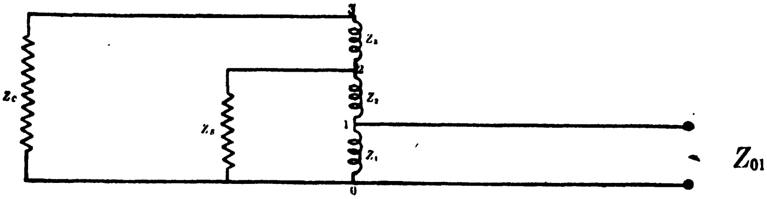

Fig. 7C. Impedance "looking into" the 0–1 terminals of an ideal three-winding auto-transformer.

$$Z_{01} = \frac{Z_B Z_C Z_1}{Z_B(\sqrt{Z_1} + \sqrt{Z_2} + \sqrt{Z_3})^2 + Z_C(\sqrt{Z_1} + \sqrt{Z_2})^2} \tag{35}$$

V. Impedance Relations in Ideal Transformers Having Four or More Windings.

The impedance looking into any winding of an ideal transformer having four or more separate windings may be derived in a way similar to that employed in obtaining equation (21). For example, consider the five-winding ideal transformer shown in Fig. 8:

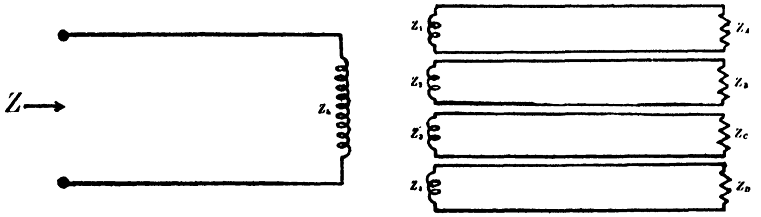

FIG. 8. Impedance "looking into" one winding of an ideal transformer having five separate windings.

$$Z = \frac{Z_5 Z_A Z_B Z_C Z_D}{Z_1(Z_B Z_C Z_D) + Z_2(Z_A Z_C Z_D) + Z_3(Z_A Z_B Z_D) + Z_4(Z_A Z_B Z_C)} \tag{36}$$

Similarly, the impedance looking into any winding of a multi-winding ideal auto-transformer may be obtained, by analogy, from equations (33), (34) and (35). For example, consider the transformers of Fig. 9, in which the windings are assumed to be in a series aiding connection. The impedances indicated in Figs. 7A, 7B and 7C are given by equations (37), (38) and (39), respectively.

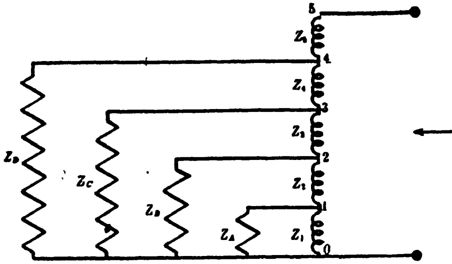

Fig 9A

$$Z_{05}=\frac{Z_AZ_BZ_CZ_D(\sqrt{Z_1}+\sqrt{Z_2}+\sqrt{Z_3}+\sqrt{Z_4}+\sqrt{Z_5})^2}{Z_1Z_BZ_CZ_D+(\sqrt{Z_1}+\sqrt{Z_2})^2(Z_AZ_CZ_D)+(\sqrt{Z_1}+\sqrt{Z_2}+\sqrt{Z_3})^2(Z_AZ_BZ_D)+(\sqrt{Z_1}+\sqrt{Z_2}+\sqrt{Z_3}+\sqrt{Z_4})^2(Z_AZ_BZ_C)} \tag{37}$$

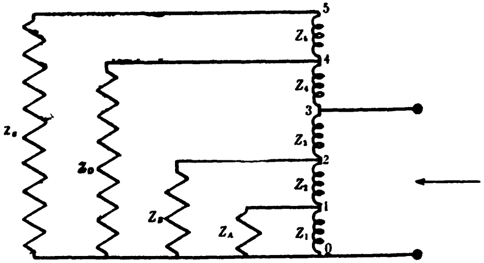

Fig 9B

$$Z_{03}=\frac{Z_AZ_BZ_DZ_E(\sqrt{Z_1}+\sqrt{Z_2}+\sqrt{Z_3})^2}{Z_1Z_BZ_DZ_E+(\sqrt{Z_1}+\sqrt{Z_2})^2(Z_AZ_DZ_E)+(\sqrt{Z_1}+\sqrt{Z_2}+\sqrt{Z_3}+\sqrt{Z_4})^2(Z_AZ_BZ_E)+(\sqrt{Z_1}+\sqrt{Z_2}+\sqrt{Z_3}+\sqrt{Z_4}+\sqrt{Z_5})^2(Z_AZ_BZ_D)} \tag{38}$$

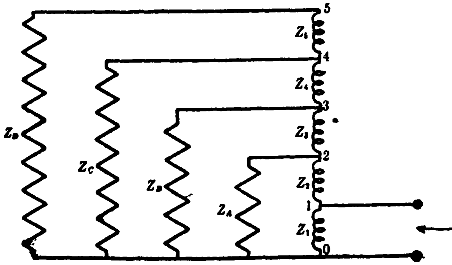

Fig. 9C

$$Z_{01}=\frac{Z_AZ_BZ_CZ_DZ_1}{(\sqrt{Z_1}+\sqrt{Z_2})^2Z_BZ_CZ_D+(\sqrt{Z_1}+\sqrt{Z_2}+\sqrt{Z_3})^2Z_AZ_CZ_D+(\sqrt{Z_1}+\sqrt{Z_2}+\sqrt{Z_3}+\sqrt{Z_4})^2Z_AZ_BZ_D+(\sqrt{Z_1}+\sqrt{Z_2}+\sqrt{Z_3}+\sqrt{Z_4}+\sqrt{Z_5})^2Z_AZ_BZ_C} \tag{39}$$

FIG. 9. Impedance "looking into" various terminals of an ideal five-winding auto-transformer. Fig. 9A. Impedance at terminals 0–5. Fig. 9B. Impedance at terminals 0–3. Fig. 9C. Impedance at terminals 0–1.

LIST OF THE MORE IMPORTANT SYMBOLS USED IN THE TEXT

a	Ratio of frequency of infinite attenuation to the cut-off frequency, f_∞/f_c or f_c/f_∞.
a'	Constant used in low pass and high pass wave filter sections having mutual inductance.
A	Real component of the propagation constant, P. In the case of a symmetrical structure, it is the *attenuation constant.*
A	Natural attenuation units, napiers, or hyps.
A_C	Attenuation constant of standard cable.
A_m	Reflection loss, in napiers, when filter is terminated in a mid-section.
A_O	Phase difference loss, in napiers, or $A_{\text{TRT.}} - A_{\text{TRF.}}$.
$A_{\text{TRF.}}$	Transformer loss, in napiers.
$A_{\text{TRT.}}$	Transition loss, in napiers.
A_x	Total terminal loss, in napiers, when filter is terminated in an x-series end section. $A_x = A_x' + A_x''$.
A_x'	Reflection loss, in napiers, when filter is terminated in a one-half section.
A_x''	Transmission loss, in napiers, due to a $(x - .5)$ series termination.
B	Imaginary component of the propagation constant, P. In the case of a *symmetrical* structure, it is the *phase constant.*
C	Capacity or capacitance.
e	Electromotive force.
ϵ	Natural base, 2.718.
E	Electromotive force.
f	Frequency.
f_c	Cut-off frequency of a low or high pass wave filter.
f_∞	Frequency of infinite attenuation.
f_1	Lower cut-off frequency of a band pass or band elimination wave filter.
f_2	Higher cut-off frequency of a band pass or band elimination wave filter.

F	Constant used in low pass and high pass wave-filter design.
G	Leakance.
G	Constant used in low pass and high pass wave-filter design.
H	Denominator of fractions.
H	Constant used in low pass and high pass wave-filter design.
I	Current (used with various self-explanatory subscripts).
i	The quadrantal operator, $\sqrt{-1}$.
k	$\lvert Z_1/4Z_2 \rvert = \sqrt{U^2 + V^2}$.
K	Ratio of the reactance of a receiver to its effective resistance. See also Q.
K	Constant. Also an integer.
K	Coefficient of coupling.
l	Inductance.
L	Inductance.
L	Loss in miles of standard cable.
L	Length.
L_O	Argument loss in miles of standard cable, or $L_{\text{TRT.}} - L_{\text{TRF}}$
$L_{\text{TRF.}}$	Transformer loss, in miles of standard cable.
$L_{\text{TRT.}}$	Transition loss, in miles of standard cable.
m	Constant employed in derived wave-filter types.
N	Any number.
N	Number of turns (used with various self-explanatory subscripts).
N_A	Number of attenuation units or napiers.
$N_{\text{M.S.C.}}$	Number of miles of standard cable.
N_{TU}	Number of transmission units.
P	Propagation constant.
P'	Propagation constant of a loaded circuit.
Q	Coil dissipation constant, or the ratio of the reactance of a coil to its effective resistance, $L\omega/R$.
r	Ratio of any two quantities.
r	Effective resistance.
R	Effective resistance (used with various self-explanatory subscripts).
R_L	Effective resistance of Z_L.
R_O	Effective resistance of Z_O.
R_R	Effective resistance of Z_R.
R_S	Effective resistance of Z_S.
R_T	Effective resistance of Z_T.
R_{11}	Effective resistance of Z_{11}.

R_{21}	Effective resistance of Z_{21}.
t	Time, in seconds.
U	Real component in generalized ladder structure.
U_K	Real component in a constant-K structure.
V	Velocity of propagation.
V	Imaginary component in generalized ladder structure.
V_K	Imaginary component in a constant K structure.
W	Power or energy.
x	Reactance.
x	Fractional part of the series impedance, Z_1, of a ladder type of structure.
X	Reactance (used with various self-explanatory subscripts).
X_L	Reactance of Z_L.
X_O	Reactance of Z_O.
X_R	Reactance of Z_R.
X_S	Reactance of Z_S.
X_T	Reactance of Z_T.
X_{11}	Reactance of Z_{11}.
X_{21}	Reactance of Z_{21}.
Y	Power-distribution ratio.
z	Impedance.
Z	Impedance (used with various self-explanatory subscripts). Usually employed as a vector quantity.
$\bar{Z}$	Conjugate of Z, i.e., if $Z = R + jX$, $Z = R - jX$.
Z_c	Impedance of a loading coil.
Z_C	Any series impedance.
Z_{I_1}	Image impedance at one end.
Z_{I_2}	Image impedance at other end.
Z_k	Iterative impedance, of structure with smoothly distributed constants.
Z_K	Iterative impedance, mid-series, or mid-coil.
Z_K'	Iterative impedance, mid-shunt, or mid-section.
Z_{K_1}	Iterative impedance, at one end.
Z_{K_2}	Iterative impedance, at other end.
Z_L	Impedance of a line.
Z_M	Mutual impedance.
Z_O	Impedance of structure, with distant end open-circuited.
Z_O	Nominal iterative impedance.

Z_P	Self impedance of a primary winding.
Z_R	Impedance of any receiving device or circuit.
Z_S	Self impedance of a secondary winding.
Z_S	Impedance of structure, with distant end short-circuited.
Z_S	Shunt impedance.
Z_T	Impedance of any transmitting device or circuit.
Z_1	Total series impedance—between shunt arms—of generalized ladder type of structure.
Z_1	Any impedance.
Z_2	Total shunt impedance—between series arms—of generalized ladder type of structure.
Z_2	Any impedance.
Z_{11}, Z_{12}, Z_{13}, etc.	Impedances in a series arm.
Z_{21}, Z_{22}, Z_{23}, etc.	Impedances in a shunt arm.
Z_{12}, Z_{AB}, etc.	Mutual impedances.
α	The real component of the image transfer constant, or the attenuation constant.
α	Angle.
β	The imaginary component of the image transfer constant, or the phase constant.
β	The difference between two angles or $\theta - \phi$.
δ	A relatively small quantity.
Δ	Increment.
ϵ	Natural base, 2.718.
θ	Angle.
Θ	Image transfer constant.
Θ	Angle.
λ	Wave-length.
μ	Voltage amplification constant.
π	3.1416.
ϕ	Angle.
ω	$2\pi f$.
$=$	Equals.
$\neq$	Is not equal to.
$\equiv$	Equals by definition.
$\doteq$	Equals approximately.
$\lvert\ \rvert$	Absolute or scalar value (modulus) of the complex quantity enclosed.
$/\underline{\quad}$	Sign of a positive angle.
$/\overline{\quad}$	Sign of a negative angle.

$<$	Is less than.
$>$	Is greater than.
$\sim$	Cycles per second.
!	Factorial.

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